* Discuss the key mistakes from Midterm 2.

* Do Ex 4 on Stokes' Theorem from Lecture #22

* Today: Triple integrals (section 15.6 of the textbook).

This is going to be completely analogously to double integrals, but a bit harder.

Want: Integrate a function $f(x,y,z)$ over a solid $E \subset \mathbb{R}^3$.

As in the case of double integrals, we start from the simplest case of solids

$$\boxed{B = [a,b] \times [c,d] \times [r,s] = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\}}$$

rectangular box

Fubini's Theorem: If $f(\cdot,\cdot,\cdot)$ is a continuous function on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz}$$

! Note that there are 6 possibilities for the right-hand side, depending on the order of dx, dy, dz in the iterated integral.

Ex1: Evaluate the triple integral $\iiint_B 6xye^z\,dV$, $B = [-1,2] \times [0,2] \times [0,1]$.

$$\iiint_B 6xye^z\,dV = \int_0^1 \int_0^2 \int_{-1}^2 6xye^z\,dx\,dy\,dz = \int_0^1 \int_0^2 \left[(3x^2ye^z)\Big|_{x=-1}^{x=2} \right] dy\,dz = \int_0^1 \left[\left(\frac{9}{2}e^z y^2\right)\Big|_{y=0}^{y=2} \right] dz$$

$$= \int_0^1 18e^z\,dz = 18(e-1)$$

Lecture #23 11/27/2018

Similarly to the double integrals, we can define the triple integral of $f(\cdot,\cdot,\cdot)$ over a general bounded region E in $\mathbb{R}^3$ by reducing to previous case. Pick a rectangular box $B \subseteq \mathbb{R}^3$ containing E, and define function F on B as follows:

$$F(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in E \\ 0 & \text{if } (x,y,z) \in B \setminus E. \end{cases}$$

Then we define

$$\boxed{\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV}$$

On the technical side, this means that as we represent this triple integral by an iterated integral, the limits of integration are no longer fixed.

Example 1: If $E = \{(x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$, then

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA}$$

Here D is the projection of E onto yz-plane.

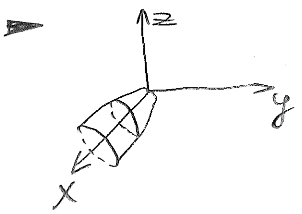
If we further e.g. can describe D as $D = \left\{(y,z) \mid \begin{array}{l} a \le z \le b \\ g_1(z) \le y \le g_2(z) \end{array}\right\}$, then we can further expand:

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(z)}^{g_2(z)} \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\,dy\,dz}$$

Example 2: Completely likewise, if $E = \left\{(x,y,z) \,\middle|\, \begin{array}{l} a \le x \le b,\ g_1(x) \le y \le g_2(x), \\ u_1(x,y) \le z \le u_2(x,y) \end{array}\right\}$, then

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

Ex2: Evaluate $\iiint\limits_E \sqrt{y^2+z^2}\,dV$, where E is the region bounded by the paraboloid $x=y^2+z^2$ and the plane $x=9$.

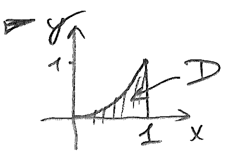


$$\iiint\limits_E \sqrt{y^2+z^2}\,dV = \iint\limits_D \left(\int_{y^2+z^2}^{9} \sqrt{y^2+z^2}\,dx \right) dA,$$ where D is the disk $\{(y,z) \mid y^2+z^2 \le 9\}$

$$= \iint\limits_D (9-y^2-z^2)\sqrt{y^2+z^2}\,dA \xrightarrow[\substack{y=r\cos\theta \\ z=r\sin\theta}]{\text{Polar}} \int_0^{2\pi}\int_0^3 (9-r^2)\cdot r \cdot r \cdot dr\,d\theta =$$

$$= \int_0^{2\pi} \left(\frac{9r^3}{3} - \frac{r^5}{5} \right) \Big|_{r=0}^{r=3} d\theta = \left(81 - \frac{243}{5}\right)\cdot 2\pi = \boxed{\frac{324}{5}\pi}$$

Ex3: Evaluate the triple integral $\iiint\limits_E xy\,dV$, where E lies under the plane $z=2+x+y$ and above the region in the xy-plane bounded by $y=x^2$, $y=0$, $x=1$.



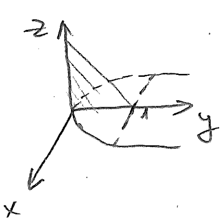
$$\iiint\limits_E xy\,dV = \iint\limits_D \left(\int_0^{2+x+y} xy\,dz \right) dA = \int_0^1 \int_0^{x^2} \int_0^{2+x+y} xy\,dz\,dy\,dx$$

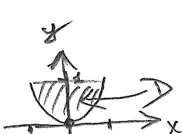
$$= \int_0^1 \int_0^{x^2} (2xy + x^2y + xy^2)\,dy\,dx = \int_0^1 \left[\left(xy^2 + \frac{x^2y^2}{2} + \frac{xy^3}{3} \right) \Big|_{y=0}^{y=x^2} \right] dx$$

$$= \int_0^1 \left(x^5 + \frac{x^6}{2} + \frac{x^7}{3} \right) dx = \boxed{\frac{1}{6} + \frac{1}{14} + \frac{1}{24}}$$

Ex 4: Find the Volume of the solid E enclosed by the cylinder $y=x^2$ and the planes $z=0$, and $y+z=1$.

$$\mathrm{Vol}(E) = \iiint\limits_E 1\,dV = \iint\limits_D \left(\int_0^{1-y} 1\,dz \right) dA = \iint\limits_D (1-y)\,dA = \int_{-1}^{1} \int_{x^2}^{1} (1-y)\,dy\,dx$$





$$= \int_{-1}^{1} \left(y - \frac{y^2}{2} \right) \Big|_{y=x^2}^{y=1} dx = \int_{-1}^{1} \left(\frac{1}{2} - x^2 + \frac{x^4}{2} \right) dx = \left(\frac{1}{2}x - \frac{1}{3}x^3 + \frac{1}{10}x^5 \right) \Big|_{x=-1}^{x=1}$$

$$= \boxed{1 - \frac{2}{3} + \frac{1}{5}}$$