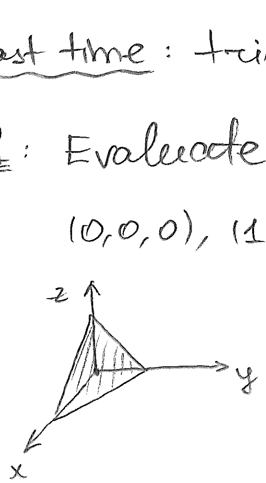
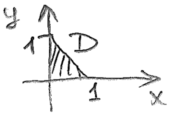
Lecture #24

* Last time: triple integrals.

Ex1: Evaluate $\iiint_E 6xy\,dV$, where E is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.



$$\iiint_E 6xy\,dV = \iint_D \left(\int_0^{1-x-y} 6xy\,dz\right)dA,$$ where D is the projection of E onto xy-plane 

, while upper bound $1-x-y$ is coming from the fact that the equation of the plane through $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ is $x+y+z=1$.

So: $$\iiint_E 6xy\,dV = \iint_D \left(\int_0^{1-x-y} 6xy\,dz\right)dA = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 6xy\,dz\,dy\,dx =$$

$$= \int_0^1 \int_0^{1-x} 6xy(1-x-y)\,dy\,dx = \int_0^1 \left[(6x-6x^2)\cdot\frac{y^2}{2} - 6x\cdot\frac{y^3}{3}\right]\Bigg|_{y=0}^{y=1-x} dx =$$

$$= \int_0^1 \left((6x-6x^2)\cdot\frac{(1-x)^2}{2} - 6x\cdot\frac{(1-x)^3}{3}\right)dx = \int_0^1 \left[(3x-3x^2)(1-2x+x^2) - 2x(1-3x+3x^2-x^3)\right]dx =$$

$$= \int_0^1 (x-3x^2+3x^3-x^4)\,dx = \frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}$$

Physical Meaning

(1) $\iiint_E 1\,dV$ is just the volume of E.

(2) If $\rho(x,y,z)$ is the density function of a solid object occupying region E, then $m = \iiint_E \rho(x,y,z)\,dV$ is just its mass.

(3) Likewise $\left(\frac{1}{m}\iiint_E x\rho(x,y,z)\,dV,\ \frac{1}{m}\iiint_E y\rho(x,y,z)\,dV,\ \frac{1}{m}\iiint_E z\rho(x,y,z)\,dV\right)$ are the coordinates of the center of mass.

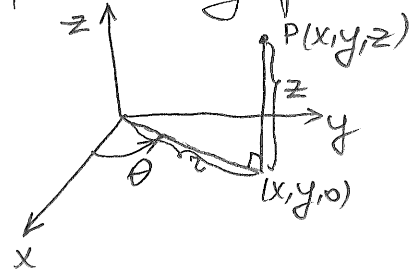
Ex2: Describe the solid whose volume is given by
$$\int_0^2 \int_0^{1-\frac{x}{2}} \int_0^{3-z} dy\,dz\,dx.$$

11/29/2018

Lecture #24

* Cylindrical Coordinates

Analogously to polar coordinates in $\mathbb{R}^2$, one may work with cylindrical coordinate system in $\mathbb{R}^3$, where a point P is represented by (r, θ, z); here z – usual z-coordinate, while (r,θ) are polar coordinates of the projection of P onto xy-plane.



Note: • $x = r\cos\theta, y = r\sin\theta, z = z$.

• $r = \sqrt{x^2+y^2}$, $\tan(\theta) = \frac{y}{x}$, $z = z$

Ex3: (a) Describe the surface whose equation in cyl. coordinates is $z = 2r$.

(b) –//– $z^2 + r^2 = 9$.

◤ (a) $z = 2r \Leftrightarrow z = 2\sqrt{x^2+y^2}$, which determines a cone.

(b) $z^2 + r^2 = 9 \Leftrightarrow z^2 + x^2 + y^2 = 9$, which determines a sphere ■

* Triple Integrals in Cylindrical coordinates

Suppose that $E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$ and D is conveniently described in polar coordinates as

$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$.

Then, switching $(x,y) \rightsquigarrow (r,\theta)$ in $\iiint_E f(x,y,z) = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$,

we get the formula for triple integration in cylindrical coordinates:

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) \cdot r\,dz\,dr\,d\theta}$$

Ex 4: Evaluate $\iiint_E \sqrt{x^2+y^2}\, dV$, where E is the region inside the cylinder $x^2+y^2=4$ and between the planes $z=-1$ and $z=2$.

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^2\int_{-1}^2 r^2\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 3r^2\,dr\,d\theta = \int_0^{2\pi} 8\,d\theta = \boxed{16\pi}$$

Ex 5: Find the volume of the solid E enclosed by the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=2$.

The intersection of the specified cone and sphere is a circle $\{(x,y,1) \mid x^2+y^2=1\}$. $\Rightarrow$ projection D of E onto xy-plane is the unit disc.

$$Vol(E) = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(r\sqrt{2-r^2}-r^2\right)dr\,d\theta =$$

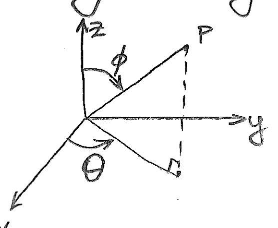
$$= 2\pi\cdot\left(\int_0^1 r\sqrt{2-r^2}\,dr - \frac{r^3}{3}\Big|_{r=0}^{r=1}\right) = 2\pi\left(\int_0^1 r\sqrt{2-r^2}\,dr - \frac{1}{3}\right)$$

$$u=2-r^2 \Rightarrow \int_0^1 r\sqrt{2-r^2}\,dr = \int_2^1 \sqrt{u}\cdot\frac{du}{-2} = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_{u=1}^{u=2} = \frac{2\sqrt{2}-1}{3}$$

So: $Vol(E) = 2\pi\left(\frac{2\sqrt{2}-1}{3} - \frac{1}{3}\right) = \frac{4\pi(\sqrt{2}-1)}{3}$

* Spherical Coordinates

Recall that the spherical coordinates (ρ,θ,ϕ) of a point P in $\mathbb{R}^3$ are given by the following picture:



Note: $\rho \geq 0$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$.

$$\boxed{x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi}$$

Other way around: $\rho=\sqrt{x^2+y^2+z^2}$, $\phi=\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$, $\tan\theta=\frac{y}{x}$.

Ex 6: (a) The point $(3, \underset{=\theta}{\frac{\pi}{3}}, \underset{=\phi}{\frac{\pi}{4}})$ is given in spherical coordinates. Find its x,y,z-coord.

(b) Find spherical coordinates of the point $(3,0,4)$ given in x,y,z-coord.

(a) $x=\frac{3\sqrt{2}}{4}$, $y=\frac{3\sqrt{6}}{4}$, $z=\frac{3\sqrt{2}}{2}$

(b) $\rho=5$, $\theta=0$, $\phi=\cos^{-1}(4/5)$

11/29/2018

Lecture #24

* Triple integrals in spherical coordinates

First, consider a counterpart of the rectangular box in spherical coordinates, a.k.a. "spherical wedge":

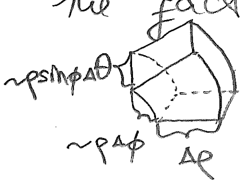
$$\boxed{E = \{(\rho, \theta, \phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}}$$

$a \ge 0$
$\beta - \alpha \le 2\pi$
$d - c \le \pi$

Then the triple integral in spherical coordinates:

$$\boxed{\iiint_E f(x,y,z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho\cos\phi) \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi}$$

Remark 1: The extra factor $\rho^2 \sin\phi$ originates for the same reason as the factor r in polar change:



Volume $\sim (\Delta\rho)\cdot(\rho\Delta\phi)\cdot(\rho\sin\phi\,\Delta\theta) = \rho^2 \sin\phi \cdot \Delta\rho\,\Delta\phi\,\Delta\theta$.

Remark 2: If the region E is more complicated, we will have the same formula, but the limits of inner integration will not be constants! In other words, if $E = \{(\rho,\theta,\phi) \mid \alpha \le \theta \le \beta,\ c \le \phi \le d,\ g_1(\theta,\phi) \le \rho \le g_2(\theta,\phi)\}$, then we should take $\int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} \ldots$

Ex 7: Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{\frac{x^2+y^2}{3}}$ and below the sphere $x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}$.

The eq-n of the cone: $\rho\cos\phi = \frac{\rho\sin\phi}{\sqrt{3}} \Leftrightarrow \tan\phi = \sqrt{3} \Leftrightarrow \phi = \frac{\pi}{3}$

The eq-n of the sphere: $x^2+y^2+z^2 = z \Leftrightarrow \rho^2 = \rho\cos\phi \Leftrightarrow \rho = \cos\phi$.

Thus: $E = \{(\rho,\theta,\phi) \mid 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{3},\ 0 \le \rho \le \cos\phi\}$

So: $Vol(E) = \iiint_E dV = \int_0^{\pi/3} \int_0^{2\pi} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi = \frac{2\pi}{3} \int_0^{\pi/3} \cos^3\phi \cdot \sin\phi\, d\phi$

$\underline{u = \cos\phi}$ $\frac{2\pi}{3}\int_1^{1/2} u^3(-du) = \frac{2\pi}{3}\cdot \frac{u^4}{4}\Big|_{u=1/2}^{u=1} = \boxed{\frac{\pi}{6}\left(1 - \frac{1}{16}\right)}$