Lecture # 25

12/04/2018

* Last time: Cylindrical and Spherical coordinates and triple integrals via those.

First, let us recall the formulas we had:

Cylindrical Coordinates

Suppose that $E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$ and

$D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$ in polar coordinates.

Then, the triple integration in cylindrical coordinates looks as:

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\cdot r\,dz\,dr\,d\theta}$$

Spherical coordinates

Assume E is given in spherical coordinates by

$E = \{(\rho, \theta, \phi) \mid c \leq \phi \leq d,\ \alpha \leq \theta \leq \beta,\ g_1(\theta,\phi) \leq \rho \leq g_2(\theta,\phi)\}$.

Then, the triple integration in spherical coordinates looks as:

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\cdot \rho^2\sin\phi\,d\rho\,d\theta\,d\phi}$$

Ex1: (a) Set up the integral $\iiint_B e^{\frac{1}{3}(x^2+y^2+z^2)^{3/2}}\,dV$ in rectangular coordinates, where $B = \{(x,y,z) \mid x^2+y^2+z^2 \leq 4\}$ is the ball of radius 2. Can you compute?

(b) Same question, but in cylindrical coordinates.

(c) Same question, but in spherical coordinates.

(a) $$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} e^{\frac{1}{3}(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx$$

(b) $$\int_0^{2\pi} \int_0^{2} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} e^{\frac{1}{3}(r^2+z^2)^{3/2}}\cdot r\,dz\,dr\,d\theta$$

(c) $$\int_0^{\pi} \int_0^{2\pi} \int_0^{2} e^{\frac{1}{3}\rho^3}\cdot \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\int_0^{2\pi} \sin\phi\cdot\left(e^{\frac{1}{3}\rho^3}\Big|_{\rho=0}^{\rho=2}\right)d\theta\,d\phi = \boxed{2\cdot 2\pi\cdot(e^{\frac{8}{3}}-1)}$$

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Ex2: Use spherical coordinates to find the volume of the solid that lies above the cone $z=\sqrt{\frac{x^2+y^2}{3}}$ and below the sphere $x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}$.

- The equation of the cone: $\rho\cos\phi = \frac{\rho\sin\phi}{\sqrt{3}} \Leftrightarrow \tan(\phi)=\sqrt{3} \Rightarrow \phi=\frac{\pi}{3}$
- The equation of the sphere: $x^2+y^2+z^2=z \Leftrightarrow \rho^2=\rho\cos\phi \Leftrightarrow \rho=\cos\phi$.

So: The solid E in spherical coordinates is given by
$$E=\{(\rho,\theta,\phi)\mid 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{3},\ 0\le\rho\le\cos\phi\}$$

Hence: $\mathrm{Vol}(E)=\iiint_E 1\,dV=\int_0^{\pi/3}\int_0^{2\pi}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\frac{2\pi}{3}\int_0^{\pi/3}\cos^3\phi\sin\phi\,d\phi=$

$$\overset{u=\cos\phi}{=\!=}\ \frac{2\pi}{3}\int_1^{1/2}u^3\cdot(-du)=\frac{2\pi}{3}\int_{1/2}^{1}u^3\,du=\frac{2\pi}{3}\cdot\left(\frac{1}{4}u^4\right)\Big|_{u=1/2}^{u=1}=\boxed{\frac{\pi}{6}\left(1-\frac{1}{16}\right)}$$ ∎

Ex3: Setup the integrals evaluating $\iiint_B f(x,y,z)\,dV$.

(a) B – the "inside" part of the cone $z=\sqrt{\frac{x^2+y^2}{3}}$ bounded from above by the plane Π containing the curve of intersection of this cone and $x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}$. Use both spherical and cylindrical coordinates.

(b) B – part of the ball $x^2+y^2+(z-\frac{1}{2})^2\le\frac{1}{4}$ lying above plane Π from (a). Use both spherical and cylindrical coordinates.

(c) B – part of the ball $x^2+y^2+(z-\frac{1}{2})^2\le\frac{1}{4}$ which is outside the cone $z\ge\sqrt{\frac{x^2+y^2}{3}}$. Setup both in spherical and cylindrical coordinates.

(a) As computed in problem Ex2, the angle of the cone is $\phi=\pi/3$. Let us now find the "height" of the plane Π which is parallel to xy-plane (this is clear from the picture).

$$\left.\begin{array}{l} x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}\Leftrightarrow x^2+y^2+z^2=z \\ z=\sqrt{\frac{x^2+y^2}{3}}\Leftrightarrow x^2+y^2=3z^2\ \&\ z\ge 0\end{array}\right\}\Rightarrow \text{for points of intersection}$$

$4z^2=z\Rightarrow z=1/4$ (if $z=0\Rightarrow x=y=0$ $\Rightarrow$ not on the plane)

So: The plane Π is given by $z=\frac{1}{4}$

↓

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(Continuation of Ex3)

(a) Thus in spherical coordinates, we get (as $z=\rho\cos\phi \Rightarrow \rho = z/\cos\phi$):

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_0^{\pi/3}\int_0^{2\pi}\int_0^{1/4\cos\phi} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\theta\, d\phi}$$

In cylindrical coordinates, we get:

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}/4}\int_{r/\sqrt{3}}^{1/4} f(r\cos\theta, r\sin\theta, z)\cdot r\, dz\, dr\, d\theta}$$

Note: • the upper bound $\frac{\sqrt{3}}{4}$ for r comes as follows: $r=\sqrt{x^2+y^2}\le\sqrt{3}\cdot z\le\frac{\sqrt{3}}{4}$.

• the lower bound $r/\sqrt{3}$ for z comes from $z=r/\sqrt{3}$ on the cone.

(b) In part (b), note that ρ ranges from $1/4\cos\phi$ (at points on π) to $\cos\phi$ (at points on the sphere), hence, in spherical coordinates, we get

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_0^{\pi/3}\int_0^{2\pi}\int_{1/4\cos\phi}^{\cos\phi} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\theta\, d\phi}$$

In cylindrical coordinates z ranges from $1/4$ (only if $r\le\sqrt{3}/4$) up to the larger solution of $r^2+(z-\frac{1}{2})^2=\frac{1}{4}$, that is, up to $\frac{1}{2}+\sqrt{\frac{1}{4}-r^2}$. So, in cylindrical coordinates:

$$\boxed{\begin{aligned}\iiint_B f(x,y,z)\,dV &= \int_0^{2\pi}\int_0^{\sqrt{3}/4}\int_{1/4}^{\frac{1+\sqrt{1-4r^2}}{2}} f(r\cos\theta, r\sin\theta, z)\cdot r\, dz\, dr\, d\theta\, + \\ &+ \int_0^{2\pi}\int_{\sqrt{3}/4}^{1/2}\int_{\frac{1-\sqrt{1-4r^2}}{2}}^{\frac{1+\sqrt{1-4r^2}}{2}} f(r\cos\theta, r\sin\theta, z)\cdot r\, dz\, dr\, d\theta\end{aligned}}$$

(c) In spherical coordinates:

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_{\pi/3}^{\pi/2}\int_0^{2\pi}\int_0^{\cos\phi} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\cdot\rho^2\sin\phi\, d\rho\, d\theta\, d\phi}$$

In cylindrical coordinates:

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}/4}\int_{\frac{1-\sqrt{1-4r^2}}{2}}^{1/4} f(r\cos\theta, r\sin\theta, z)\cdot r\, dz\, dr\, d\theta}$$

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Ex 4 (# 15.8.43 from textbook): Evaluate the integral (by changing to spherical coord.)

$$\boxed{\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{3/2}\, dz\,dy\,dx}$$

This integral equals $\iiint_E (x^2+y^2+z^2)^{3/2}\, dV$ for the corresponding solid E.

Let us visualize the solid E.

- $-2 \le x \le 2$
- $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$\Rightarrow$ this means that the projection of E onto xy-plane is the disk of radius 2 centered at (0,0)

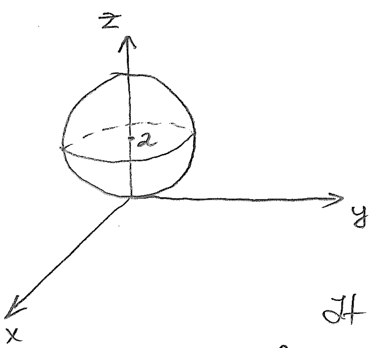
Now given (x,y), we see that $2-\sqrt{4-x^2-y^2} \le z \le 2+\sqrt{4-x^2-y^2}$

$$\Updownarrow$$

$$-\sqrt{4-x^2-y^2} \le z-2 \le \sqrt{4-x^2-y^2}$$

$$\Updownarrow$$

determines a ball of radius 2. centered at (0,0,2). $\longrightarrow$ $\boxed{x^2+y^2+(z-2)^2 \le 4}$



It is clear that it will simplify if we rewrite the above triple integral in spherical coordinates

1) clear that $0 \le \theta \le 2\pi$
2) also clear that ϕ ranges from 0 to $\pi/2$.
3) finally to find the maximal value of ρ given (ϕ,θ), write eq-n of the sphere as $x^2+y^2+z^2 = 4z \Rightarrow \rho^2 = 4\rho\cos\phi \Rightarrow \rho = 4\cos\phi$.

$$\Rightarrow \text{get } \int_0^{\pi/2}\int_0^{2\pi}\int_0^{4\cos\phi} \rho^3\cdot\rho^2\sin\phi\, d\rho\, d\theta\, d\phi = \int_0^{\pi/2}\int_0^{2\pi}\left(\frac{\rho^6}{6}\cdot\sin\phi\right)\Big|_{\rho=0}^{\rho=4\cos\phi} d\theta\, d\phi =$$

$$= 2\pi\cdot\frac{4^6}{6}\cdot\int_0^{\pi/2}\cos^6\phi\sin\phi\, d\phi \overset{u=\cos\phi}{=\!=} \frac{4^6}{3}\cdot\pi\cdot\int_1^0 u^6(-du) = \frac{4^6}{3\cdot 7}\pi = \boxed{\frac{4096\pi}{21}}$$

■

! There are two similar problems on the homework.

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* Last topic in the course: Divergence Theorem (Section 16.9 of the textbook)

Divergence Theorem: Let E be a bounded closed solid region in $\mathbb{R}^3$ with a piece-wise smooth boundary S endowed with an outward orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region containing E. Then:

$$\boxed{\iint_S \vec{F}\, dS = \iiint_E \operatorname{div}(\vec{F})\, dV}$$

Ex 5: Find the flux of the vector field $\vec{F} = \langle y + e^{z^2}, -\sin(x^3) + z^{10}, 3z - x^{200}y^{100}\rangle$ over the sphere $x^2+y^2+z^2=4$ oriented inwards

- $\operatorname{div}(\vec{F}) = 0+0+3 = 3$
- Clearly S is the boundary of the ball $B: x^2+y^2+z^2 \le 4$.

So: $\iint_S \vec{F}\, dS = \underbrace{-}_{\text{opposite orientation}} \iiint_B \operatorname{div}(\vec{F})\, dV = -3 \cdot \operatorname{Vol}(B) = -3 \cdot \frac{4}{3}\pi \cdot 2^3 = \boxed{-32\pi}$

∎

Ex 6: Prove that the flux of $\operatorname{curl}(\vec{F})$ over any oriented surface S which is a boundary of some solid E, is ZERO.

Follows from the divergence thm and the equality $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$

∎

Ex 7: Evaluate $\iint_S \vec{F}\, dS$, $\vec{F} = \langle xy, -\frac{1}{2}y^2, z\rangle$ and the surface S (oriented outwards) consists of 3 surfaces: (1) $z = 4-3x^2-3y^2$, $1 \le z \le 4$; (2) $x^2+y^2=1$, $0 \le z \le 1$; (3) $z=0$, $x^2+y^2 \le 1$.

$\operatorname{div}(\vec{F}) = y - y + 1 = 1$

In cylindrical coordinates, the solid E bounded by S is given by $\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ 0 \le z \le 4-3r^2 \end{cases}$

So: $\iint_S \vec{F}\, dS = \iiint_E 1\, dV = \int_0^{2\pi}\int_0^1\int_0^{4-3r^2} r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 (4r - 3r^3)\, dr\, d\theta$

$= 2\pi \cdot \left(2r^2 - \frac{3}{4}r^4\right)\Big|_{r=0}^{r=1} = 2\pi \cdot \frac{5}{4} = \boxed{\frac{5\pi}{2}}$

∎

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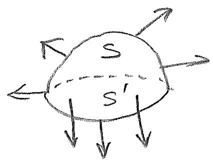
Ex 8: Evaluate the integral $\iint_S \vec{F}\, dS$, where $\vec{F} = \langle x + e^z,\ y + \sin(z^2),\ 2z+1 \rangle$ and S is the hemisphere $\begin{cases} x^2+y^2+z^2=1 \\ z \geq 0 \end{cases}$ oriented upwards.

▸ To apply the divergence theorem, we need to "close" surface S. The simplest way to achieve this is to add the disk S' on the bottom oriented "downwards"



By the divergence theorem

$$\left.\begin{aligned} &\iint_S \vec{F} dS + \iint_{S'} \vec{F} dS = \iiint_E \operatorname{div}(\vec{F})\, dV = 4 \cdot \mathrm{Vol}(E) = 4 \cdot \frac{4\pi}{3} \cdot \frac{1}{2} = \frac{8\pi}{3}. \\ &\text{Now we also need to compute } \iint_{S'} \vec{F} dS \text{ directly} \\ &\iint_{S'} \vec{F} dS = \iint_{S'} \vec{F} \cdot (-\vec{k})\, dA = \iint_{S'} (-1)\, dA = -\mathrm{Area}(S') = -\pi \end{aligned}\right\} \Rightarrow$$

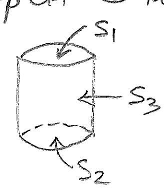
$$\Rightarrow \boxed{\iint_S \vec{F} dS = \frac{11\pi}{3}}$$

Ex 9: Verify the divergence theorem for $\vec{F} = \langle x^3, y^3, z^2 \rangle$ and E - cylindrical solid $\begin{cases} x^2+y^2 \leq 9 \\ 0 \leq z \leq 2 \end{cases}$

▸ • $\operatorname{div}(\vec{F}) = 3x^2 + 3y^2 + 2z \Rightarrow \iiint_E \operatorname{div}(\vec{F})\, dV = \int_0^{2\pi}\int_0^3\int_0^2 (3r^2 + 2z) \cdot r\, dz\, dr\, d\theta =$

$$= \int_0^{2\pi}\int_0^3 (6r^3 + 4r)\, dr\, d\theta = 2\pi \cdot \left(\frac{6}{4} r^4 + 2r^2\right)\Big|_{r=0}^{r=3} = 2\pi\left(\frac{3 \cdot 81}{2} + 18\right) = \boxed{279\pi}$$

• Split S into 3 parts:



S_1: $\vec{r}(u,v) = \langle u\cos v, u\sin v, 2 \rangle \Rightarrow \vec{r}_u \times \vec{r}_v = \langle 0, 0, u \rangle$ - looks outside!

$$\Rightarrow \iint_{S_1} \vec{F} dS = \int_0^{2\pi}\int_0^3 4u\, du\, dv = 2\pi \cdot 18 = \boxed{36\pi}$$

S_2: $\vec{r}(u,v) = \langle u\cos v, u\sin v, 0 \rangle \Rightarrow \vec{r}_u \times \vec{r}_v = \langle 0, 0, u \rangle$ - look inside $\Rightarrow$ take $-\vec{r}_u \times \vec{r}_v$

$$\Rightarrow \iint_{S_2} \vec{F} dS = -\int_0^{2\pi}\int_0^3 0 \cdot du\, dv = \boxed{0}$$

S_3: $\vec{r}(u,v) = \langle 3\cos u, 3\sin u, v \rangle \Rightarrow \vec{r}_u \times \vec{r}_v = \langle 3\cos u, 3\sin u, 0 \rangle$ - looks outwards!

$$\Rightarrow \iint_{S_2} \vec{F} dS = \int_0^{2\pi}\int_0^2 (81\cos^4 u + 81\sin^4 u)\, dv\, du = \boxed{243\pi}$$

And we see that indeed: $36\pi + 0 + 243\pi = 279\pi$ ✓ ■