Lecture #2

09/03/2019

- Office hours: Tue $2^{00}-3^{00}$, Wed $2^{30}-3^{30}$, LOM 219-C
- 1^{st} Homework → due this Thursday, Sept. 5.
  - → better start today, so that you may ask questions during office hours tomorrow
  - → Staple all pages, Write your name and Section #.

- Ask if there are any questions from our first class last week.
- Last time we discussed:
  1) computing the distance b/w 2 points in $\mathbb{R}^2$ or $\mathbb{R}^3$
  2) equation of a sphere, completing squares to find the center and the radius.
  3) vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$; addition and multiplication by scalars.
  4) components of vectors in the chosen coordinate system in $\mathbb{R}^2$ or $\mathbb{R}^3$.
     → components of $\vec{a}+\vec{b}$ and $c\cdot\vec{a}$.
  5) Unit vectors → standard basis vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$
     → unit vector $\hat{v}$ associated to any non-zero vector $\vec{v}$.
  6) Dot product: algebraic formula + geometric meaning

Let us warm up by doing a couple of examples relevant to Lecture 1.

Ex1: Find the lengths of the sides of $\triangle PQR$ with $P(2,2,2)$, $Q(4,1,1)$, $R(1,1,1)$.
Is it a right and/or isosceles triangle?

▸ $|PQ| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$, $|QR| = \sqrt{3^2+0^2+0^2} = 3$, $|PR| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.

As $|QR|^2 = |PQ|^2 + |PR|^2 \Rightarrow \triangle PQR$ is a right triangle, but clearly not an isosceles $\triangle$ ∎

Ex 2: Given $\vec{a} = 3\hat{\imath} - 2\hat{\jmath}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath}$, find $\vec{a}+2\vec{b}$ and $\|\vec{a}+2\vec{b}\|$. Also sketch $\vec{a}+2\vec{b}$.

▸ $\vec{a}+2\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 2(\hat{\imath}+2\hat{\jmath}) = 5\hat{\imath}+2\hat{\jmath}$ and $\|\vec{a}+2\vec{b}\| = \sqrt{5^2+2^2} = \sqrt{29}$ ∎

Ex3: Find the unit vector whose direction is opposite to that of $\vec{v} = \langle 1,3,2\rangle$.

▸ $-\hat{v} = -\dfrac{\langle 1,3,2\rangle}{\sqrt{14}}$ ∎

Ex 4: Find the acute angle between the lines $x+y=1$ and $x-2y=-2$

▸ The first line contains points $A(1,0)$ and $B(0,1)$, so that $\vec{AB} = \langle -1,1\rangle$

The second line contains points $C(-2,0)$ and $D(0,1)$, so that $\vec{CD} = \langle 2,1\rangle$

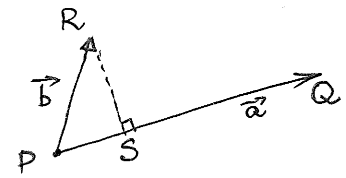
Thus: $\underbrace{\|\vec{AB}\|}_{\sqrt{2}} \cdot \underbrace{\|\vec{CD}\|}_{\sqrt{5}} \cdot \cos\underbrace{\theta}_{\text{angle between } \vec{AB} \,\&\, \vec{CD}} = \vec{AB}\cdot\vec{CD} = -1 \Rightarrow \theta = \cos^{-1}\left(-1/\sqrt{10}\right)$

Hence, the acute angle b/w the above two lines is $\cos^{-1}\left(1/\sqrt{10}\right)$ ∎

Lecture #2

* Projections

At home you should have watched the Ximera module on projections.



$\vec{PS} = \text{proj}_{\vec{a}}\vec{b}$ = vector projection of $\vec{b}$ onto $\vec{a}$

signed magnitude of $\vec{PS} = \text{comp}_{\vec{a}}\vec{b}$ = scalar projection of $\vec{b}$ onto $\vec{a}$.

Two key formulas you should have learnt from Ximera:

$$\boxed{\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}, \quad \text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a}}$$

Ex 5: Find the scalar and vector projections of $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ onto $\vec{a} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$.

▸ $\text{comp}_{\vec{a}}\vec{b} = \frac{2+2-6}{\sqrt{9}} = -2/3$

$\text{proj}_{\vec{a}}\vec{b} = \frac{2+2-6}{9}\cdot(2\hat{\imath} + \hat{\jmath} - 2\hat{k}) = -\frac{4}{9}\hat{\imath} - \frac{2}{9}\hat{\jmath} + \frac{4}{9}\hat{k}$ ∎

Ex 6*: Suppose that $\vec{a}$ and $\vec{b}$ are nonzero vectors.

(a) Under which conditions is $\text{comp}_{\vec{a}}\vec{b} = \text{comp}_{\vec{b}}\vec{a}$?

(b) — // — $\text{proj}_{\vec{a}}\vec{b} = \text{proj}_{\vec{b}}\vec{a}$?

▸ (a) $\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|} = \frac{\vec{b}\cdot\vec{a}}{\|\vec{b}\|} \iff \vec{a}\cdot\vec{b} = 0$ or $\|\vec{a}\| = \|\vec{b}\| \iff$ $\vec{a}$ and $\vec{b}$ are either orthogonal or have the same magnitude.

(b) Again evoking the formula deduce that $\vec{a} \perp \vec{b}$ or $\vec{a} = \vec{b}$
(it may be convenient to rewrite $\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}\cdot\hat{a}$) ∎

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* Cross Product

For two vectors in $\mathbb{R}^3$ (but not in $\mathbb{R}^2$), there is one more operation producing a new vector in $\mathbb{R}^3$.

Definition: Given two vectors $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$, define their cross product via

$$\boxed{\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle}$$

An easy way to remember this formula is to write down

$$\boxed{\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\imath}\cdot\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad \text{where } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{\text{definition}}{:=} ad - bc}$$

Lecture #2

Ex7: a) Find $\vec{a}\times\vec{b}$ for $\vec{a}=\langle 2,1,3\rangle$, $\vec{b}=\langle 1,2,-1\rangle$

b) Verify that $\vec{a}\times\vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$

c) Find $\vec{b}\times\vec{a}$

d) Find $\vec{a}\times\vec{a}$

a) $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 1 & 2 & -1 \end{vmatrix} = \langle -7,5,3\rangle$

b) $\langle -7,5,3\rangle\cdot\langle 2,1,3\rangle = -14+5+9=0$

$\langle -7,5,3\rangle\cdot\langle 1,2,-1\rangle = -7+10-3=0$

c) $\vec{b}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \langle 7,-5,-3\rangle$

d) $\vec{a}\times\vec{a} = \langle 0,0,0\rangle=\vec{0}$

As illustrated by the above example, the cross product always satisfies:

$\boxed{\begin{array}{l} 1)\ \vec{a}\times\vec{a}=\vec{0} \\ 2)\ \vec{b}\times\vec{a}=-\vec{a}\times\vec{b} \\ 3)\ \vec{a}\times\vec{b} \text{ is orthogonal to both } \vec{a} \text{ and } \vec{b} \end{array}}$ ← discuss why

Remark: The direction of $\vec{a}\times\vec{b}$ is given by the "right-hand rule", i.e. if the fingers of your right hand curl in the direction of a rotation ($<180°$) from $\vec{a}$ to $\vec{b}$ then your thumb points in the direction of $\vec{a}\times\vec{b}$.

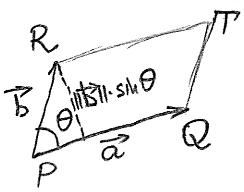
Question: What is the magnitude of $\vec{a}\times\vec{b}$?

Theorem (p. 817 of textbook): If θ is the angle between $\vec{a}$ and $\vec{b}$ $(0\le\theta\le 180°=\pi)$, then

$$\boxed{\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\cdot\|\vec{b}\|\cdot\sin\theta}$$

Corollary: $\vec{a}\times\vec{b}=0$ iff $\vec{a}$ is parallel to $\vec{b}$.

! This determines $\vec{a}\times\vec{b}$ uniquely (as we know its direction and magnitude).



$\|\vec{a}\|\cdot\|\vec{b}\|\sin\theta$ is nothing else than the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$.

So: $\boxed{\|\vec{a}\times\vec{b}\| = \text{Area (parallelogram PRTQ)} = 2\cdot\text{Area (triangle PRQ)}}$

Lecture #2

Ex 8: Find unit vectors orthogonal to the plane containing three points $P(1,2,3)$, $Q(2,1,1)$, $R(3,0,1)$.

▶ $\vec{v} := \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1,-1,-2\rangle \times \langle 2,-2,-2\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & -2 \\ 2 & -2 & -2 \end{vmatrix} = \langle -2,-2,0\rangle$

Hence the unit vectors orthogonal to that plane are

$\hat{v} = \dfrac{\langle -2,-2,0\rangle}{\sqrt{8}} = \langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\rangle$ and $-\hat{v} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle$ $\blacksquare$

Ex 9: Find the area of the triangle PQR from Ex 8.

▶ $\text{Area}(\triangle PQR) = \frac{1}{2}\,\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \sqrt{2}$ $\blacksquare$

WARNING: (1) The cross product is not commutative: $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

(2) The cross product is not associative: $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

However: For any three vectors $\vec{a}, \vec{b}, \vec{c}$ in $\mathbb{R}^3$, we have

$$\boxed{\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}}$$

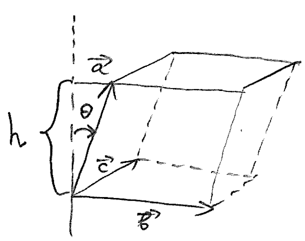
and the common value is called the scalar triple products of $\vec{a}, \vec{b}, \vec{c}$.

it is a number, not a vector

Ex 10: Prove this equality.

▶ Both expressions equal $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. Discuss details or see p. 819 of textbook $\blacksquare$

Geometric Meaning: The absolute value $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ of the scalar triple product equals the volume of the parallelepiped determined by $\vec{a}, \vec{b}, \vec{c}$.



$$\text{Vol} = \text{Area(base)} \cdot h = |\vec{b} \times \vec{c}| \cdot |\vec{a}|\cos\theta| = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

Ex 11: Verify that the vectors $\vec{a} = \langle 1,-1,-2\rangle$, $\vec{b} = \langle 2,-2,-2\rangle$, $\vec{c} = \langle -1,1,5\rangle$ are coplanar, i.e. lie in the same plane.

▶ $\vec{a}, \vec{b}, \vec{c}$ – coplanar $\iff$ Volume of parallelepiped determined by $\vec{a}, \vec{b}, \vec{c}$ is ZERO.

$$\text{Vol} = \begin{vmatrix} 1 & -1 & -2 \\ 2 & -2 & -2 \\ -1 & 1 & 5 \end{vmatrix} = 1\cdot(-10+2) - (-1)\cdot(10-2) + (-2)\cdot(2-2) = 0$$ $\blacksquare$