Lecture #3 09/05/2019

* Organization

- 1^{st} hwk should be submitted in class
- 2^{nd} hwk will be posted today
- Office Hours: Tue 2-3, Wed $2^{30}-3^{30}$, LOM 219-C

* Last time

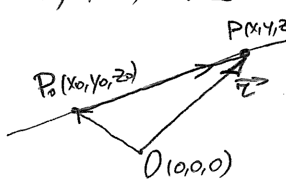
- Cross product → explicit formula
  ↳ geometric meaning

! For $\vec{a},\vec{b}$ in $\mathbb{R}^3$, the dot product $\vec{a}\cdot\vec{b}\in\mathbb{R}$ is a number, the cross product $\vec{a}\times\vec{b}$ is a vector in $\mathbb{R}^3$.

- Compare the answers to the last problem we had in class
- Discuss the scalar triple product (see the end of Lecture 2 Notes)
  ↳ computation as a determinant
  ↳ geometric meaning of its absolute value.
  ↳ Ex 11 from the Notes.

* Equations of Lines

A line L in $\mathbb{R}^3$ (similarly in $\mathbb{R}^2$) is determined uniquely once we know a point $P_0(x_0,y_0,z_0)$ on L and the direction of L, which is described by a vector $\vec{v}=\langle a,b,c\rangle$ parallel to L.



$$\vec{OP}=\underbrace{\vec{OP_0}}_{\langle x_0,y_0,z_0\rangle}+\underbrace{\vec{P_0P}}_{t\vec{v}=\langle ta,tb,tc\rangle,\ t\in\mathbb{R}}$$

So: P has coordinates $\boxed{P(x_0+ta,\ y_0+tb,\ z_0+tc)}$

Moral: (1) A vector equation of the line L is $\boxed{\vec{r}=\vec{r_0}+t\vec{v},\ t\in\mathbb{R}}$

(2) Once the coordinate system is fixed, we get a parametric equation of L:

$$\boxed{x=x_0+at,\ y=y_0+bt,\ z=z_0+ct}$$

! Warning: Each line has a lot of diferent parametric eq-s as you may:
(a) change a "starting point P_0"
(b) change the vector $\vec{v}$ by multiplying it by any number

E.g.: $(x=t,\ y=2t,\ z=3t+1)$ and $(x=1+3s,\ y=2+6s,\ z=4+9s)$ determine the same line!

Ex 1: (a) Find a vector and parametric equations of the line L passing through $P_0(2,4,1)$ and parallel to the vector $\vec{v}=\langle 1,-2,3\rangle$.

(b) Find a point on this line whose x-coordinate is ZERO.

! Def: The numbers a, b, c are called direction numbers of the line L.

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Ex 2: (a) Find parametric equation of the line that passes through $A(2,1,3)$ & $B(3,4,5)$

- Take $P_0 = A$, $\vec{v} = \vec{AB} = \langle 1,3,2 \rangle \Rightarrow$ get $(2+t, 1+3t, 3+2t)$ with $t \in \mathbb{R}$ ■

(b) Find an equation of the line segment between A and B in part (a)

- Same as in (a), but $0 \le t \le 1$ ■

! In general, the line segment between points A and B (with $\vec{OA} = \vec{r_0}$, $\vec{OB} = \vec{r_1}$) is

$$\boxed{\vec{r}(t) = \vec{r_0} + t(\vec{r_1} - \vec{r_0}) = (1-t)\vec{r_0} + t \cdot \vec{r_1}, \quad 0 \le t \le 1}$$

Ex 3: Determine whether the lines L_1, L_2 are parallel, skew, or intersecting:

L_1: $x = -2+t$, $y = 1-2t$, $z = 3t$

L_2: $x = 1-s$, $y = 2+3s$, $z = -2s$

! Warning: When working with two lines, you must use different parameters, e.g. t and s but never the same parameter t!

- $\left.\begin{array}{l} \vec{v_1} = \langle 1,-2,3 \rangle \\ \vec{v_2} = \langle -1,3,-2 \rangle \end{array}\right\}$ not parallel $\Rightarrow$ L_1 & L_2 are either skew or intersecting.

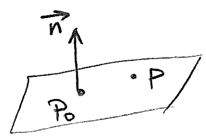
If they do intersect, there should be some t, s such that

$$\begin{cases} -2+t = 1-s \\ 1-2t = 2+3s \\ 3t = -2s \end{cases} \Leftrightarrow \begin{cases} s = 3-t \\ 1-2t = 2+3(3-t) \\ 3t = -2(3-t) \end{cases} \Leftrightarrow \begin{cases} s = 3-t \\ 1-2t = 11-3t \\ 3t = -6+2t \end{cases} \Leftrightarrow \begin{cases} s = 3-t \\ t = 10 \\ t = -6 \end{cases} \text{ not compatible!}$$

So: L_1 and L_2 must be skew. ■

* Equations of Planes

A plane in $\mathbb{R}^3$ is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\vec{n}$ that is orthogonal to the plane. $\vec{n}$ is called normal vector.



Point $P \in \mathbb{R}^3$ is in the plane iff $\vec{P_0P} \perp \vec{n} \Leftrightarrow \underbrace{(\vec{OP} - \vec{OP_0})}_{\vec{r} - \vec{r_0}} \cdot \vec{n} = 0$

So: A vector equation of the plane is $\boxed{\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0} \Leftrightarrow \boxed{\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}}$

If we fix a coordinate system and write $\vec{n} = \langle a,b,c \rangle$, then we immediately get a scalar equation of the plane through $P_0(x_0, y_0, z_0)$ and perpendicular to $\vec{n} = \langle a,b,c \rangle$

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

Finally setting $d = -ax_0 - by_0 - cz_0$, we arrive to the linear equation of the plane:

$$\boxed{ax + by + cz + d = 0}$$

Ex 4: Find an equation of the plane through the point $(3,-1,4)$ and perpendicular to the line $L: x=2-t,\ y=3+2t,\ z=1-7t$.

▸ For $\vec{n}$ we can take the direction vector of L, i.e. $\vec{n}=\langle -1,2,-7\rangle$

So: $-(x-3)+2(y+1)-7(z-4)=0$ ▫

Ex 5: Find an equation of the plane that passes through the points $P(1,2,3)$, $Q(2,1,1)$, $R(3,0,1)$.

▸ Can take $P_0=P$ and $\vec{n}=\vec{PQ}\times\vec{PR}=\langle -2,-2,0\rangle$ (see last problem in the previous class)

So: $-2(x-1)-2(y-2)=0$. ▫

Ex 6: Find the point at which the line $L: x=-1-t,\ y=2+3t,\ z=-10t$ intersects the plane $2x+y+\frac{z}{5}-1=0$

▸ Plugging expressions for x,y,z in terms of t into equation of the plane, we get:

$-2-2t+2+3t-2t-1=0 \Rightarrow -t-1=0 \Rightarrow t=-1 \Rightarrow x=-1-(-1)=0,\ y=2-3=-1,\ z=-10(-1)=10$

$\Rightarrow (0,-1,10)$ ▫

Given two planes, they are either parallel (may coincide) or intersect in a line. In the latter case, we note that:

(1) the direction of the line of intersection may be chosen as $\vec{n_1}\times\vec{n_2}$, where $\vec{n_1}$ & $\vec{n_2}$ are the normal vectors to the planes

(2) to find a point on that line, you need to find a solution of two linear equations in x,y,z.

(3) the angle between the planes is the acute angle between $\vec{n_1}$ and $\vec{n_2}$.

Ex 7: (a) Find an angle between the planes $x-y+z=2$ and $2x+y-2z=1$

(b) Find a parametric equation for the line of their intersection.

▸ (a) $\vec{n_1}=\langle 1,-1,1\rangle$, $\vec{n_2}=\langle 2,1,-2\rangle \Rightarrow \underbrace{\cos\theta}_{\text{angle b/w } \vec{n_1} \text{ and } \vec{n_2}} = \frac{\vec{n_1}\cdot\vec{n_2}}{\|\vec{n_1}\|\cdot\|\vec{n_2}\|}=\frac{-1}{3\sqrt{3}} \Rightarrow$ $\boxed{\text{angle b/w the planes is } \cos^{-1}(1/3\sqrt{3})}$

(b) $\vec{v}=\vec{n_1}\times\vec{n_2}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix}=\hat{i}+4\hat{j}+3\hat{k}=\langle 1,4,3\rangle$

To find any common solution of $\begin{cases} x-y+z=2 \\ 2x+y-2z=1 \end{cases}$ first express $x=2+y-z$ from the 1st equation and plugging into the 2nd get $4+2y-2z+y-2z=1 \Leftrightarrow 3y-4z=-3 \Rightarrow$ can pick $z=0, y=-1 \Rightarrow x=1 \Rightarrow P_0(1,-1,0)$

So: The equation of the line is

$\boxed{L: x=1+t,\ y=-1+4t,\ z=3t}$

▫

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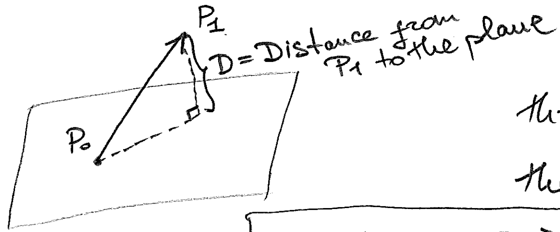
* Distance from a point to a plane



Given a point $P_1(x_1, y_1, z_1)$ and a plane passing through $P_0(x_0, y_0, z_0)$ and with a normal vector $\vec{n} = \langle a, b, c \rangle$ the distance from P_1 to this plane equals

$$\boxed{D = \left| \operatorname{comp}_{\vec{n}} \overrightarrow{P_0P_1} \right| = \frac{|\overrightarrow{P_0P_1} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}}$$

Thus, if the plane is given by a linear equation $ax + by + cz + d = 0$, then

$$\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Ex 8: Compute a distance from $P(1, 2, -3)$ to the plane $x - 2y + 3z = 8$.

▶ $D = \dfrac{|1 - 4 - 9 - 8|}{\sqrt{1 + 4 + 9}} = \dfrac{20}{\sqrt{14}}$ ∎

Note: To find the closest point on the plane to the point P_1, you just need to find the point of intersection of the plane and the line L passing through P_1 in the direction $\vec{n}$.