Lecture #4

* Last time

- Equations of the lines: $\vec{r}=\vec{r}_0+t\vec{v}$ (vector equation); $x=x_0+at,\ y=y_0+bt,\ z=z_0+ct$ (parametric equation).

Recall: Given a line by parametric equation, its direction is determined by reading off the coefficients of t, i.e. $\langle a,b,c\rangle$.

- Equations of the planes: $\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$ (vector equation); $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ (scalar equation); $ax+by+cz+d=0$ (linear equation)

- Distance formula: given a point $P_1(x_1,y_1,z_1)$ and a plane Π containing $P_0(x_0,y_0,z_0)$ and orthogonal to $\vec{n}=\langle a,b,c\rangle$, the distance D from P_1 to Π is

$$\boxed{D=\frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}}}$$

In particular if Π is given by $\underbrace{ax+by+cz+d=0}_{\text{(linear equation)}}$, then D equals

$$\boxed{D=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$$

Ex1: Compute a distance from $P(1,2,-3)$ to the plane $x-2y+3z=8$.

Plugging $a=1,\ b=-2,\ c=3,\ d=-8$ into the above formula, get $D=\frac{20}{\sqrt{14}}$ ∎

* Today: Vector functions (§13.1–13.4 in the textbook)

Def: A vector function is a function whose domain is a subset of $\mathbb{R}$ and whose range is a set of vectors

We will be mostly interested in the case of vectors in $\mathbb{R}^3$, thus a vector function is an assignment $\boxed{t\mapsto\vec{r}(t)=\langle f(t),g(t),h(t)\rangle}$ $f(t),g(t),h(t)$ – component functions

Domain is the set of t for which $\vec{r}(t)$ is defined, while range – the set of all possible $\vec{r}(t)$.

Limits & Continuity

The limit of a vector function is defined component-wise, e.g. for $\vec{r}(t)=\langle f(t),g(t),h(t)\rangle$

$$\boxed{\lim_{t\to a}\vec{r}(t)=\langle\lim_{t\to a}f(t),\ \lim_{t\to a}g(t),\ \lim_{t\to a}h(t)\rangle}$$

← if one of these limits does not exist, so is $\lim_{t\to a}\vec{r}(t)$.

Def: A vector function $\vec{r}(t)$ is continuous at $a\in\mathbb{R}$ if $\boxed{\lim_{t\to a}\vec{r}(t)=\vec{r}(a)}$

i.e. iff all components of $\vec{r}(t)$ are continuous at a.

Ex2: Find $\lim_{t\to 0}\vec{r}(t)$ for $\vec{r}(t)=\langle t+\frac{1}{t+1},\ \frac{e^t-1}{3t},\ \frac{\sin 6t}{2t}\rangle$

Use L'Hospital rule to get

$$\lim_{t\to 0}\vec{r}(t)=\langle 1,\ 1/3,\ 3\rangle$$

∎

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* Space Curves

Def: Let f, g, h be continuous $\mathbb{R}$-valued functions on an interval I. Then the locus of all points $(x,y,z)\in\mathbb{R}^3$ such that $x=f(t)$, $y=g(t)$, $z=h(t)$, $t\in I$, is called a space curve.

Note that any continuous vector function $\vec{r}(t)$ defines a space curve C, traced out by the tip of $\vec{r}(t)$.

Ex3: Describe the curves defined by the vector functions:

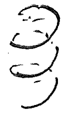
(a) $\vec{r}(t) = \langle 1+3t,\ 2-4t,\ 7+2t\rangle$

► line passing through $(1,2,7)$ and parallel to $\langle 3,-4,2\rangle$ ∎

(b) $\vec{r}(t) = \langle 2\cos t,\ 2\sin t,\ 3\rangle$

► circle of radius 2, centered at $(0,0,3)$ and parallel to the xy-plane ∎

(c) $\vec{r}(t) = \langle 2\cos t,\ 2\sin t,\ t\rangle$

►  ← helix (lies on the cylinder)

its projection to the xy-plane is a radius 2 circle, centered at $(0,0)$ ∎

Ex4: Find a vector function that represents

(a) a line segment between $P(1,2,3)$ and $Q(7,5,6)$.

► $\vec{r}(t) = \langle 1,2,3\rangle + t\langle 6,3,3\rangle = \underline{\langle 1+6t,\ 2+3t,\ 3+3t\rangle \text{ with } 0\le t\le 1}$ ∎

(b) the curve of intersection of the cylinder $x^2+y^2=4$ and the plane $x+2y+z=1$.

► Being on the cylinder, we see that (x,y) coordinates may be parametrized via $x=2\cos t$, $y=2\sin t$, $0\le t<2\pi$.

Finally, being on the plane $x+2y+z=1$ determines z via x,y: $z=1-x-2y$.

So: $\boxed{\vec{r}(t) = \langle 2\cos t,\ 2\sin t,\ 1-2\cos t-4\sin t\rangle,\ 0\le t<2\pi}$

Ex5: Hand out worksheet with matching game.

* Derivatives of vector functions

The formal definition of the derivative of a vector function $\vec{r}(t)$ may be given as for usual $\mathbb{R}$-valued functions in school:

$$\boxed{\vec{r}\,'(t) = \lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}}$$

However, this is not the way we will actually compute derivatives. Instead, we just compute component-wise, e.g. if $\vec{r}(t)=\langle f(t), g(t), h(t)\rangle$, then clearly $\vec{r}\,'(t)$ equals

$$\boxed{\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle}$$

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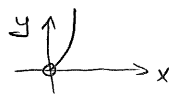
Geometric meaning: 1) From the formal definition of $\vec{r}'(t)$, it is clear that it is tangent to the space curve (determined by $\vec{r}$ at the point $\vec{r}(t)$.

2) If we think of $\vec{r}(t)$ as a coordinate-vector of the particle moving around the trajectory C, then $\|\vec{r}'(t)\|$ = speed of particle

Ex 6: (a) Sketch the plane curve with parametric equation $x = e^t, y = 2e^{3t}$

► $y = 2x^3$, $x > 0$



(b) Find $\vec{r}'(t)$.

► $\vec{r}'(t) = \langle e^t, 6e^{3t} \rangle$

(c) Find the unit tangent vector at the point with $t=0$

► $\vec{r}'(0) = \langle e^0, 6e^0 \rangle = \langle 1, 6 \rangle \Rightarrow T(0) = \dfrac{\vec{r}'(0)}{\|\vec{r}'(0)\|} = \dfrac{1}{\sqrt{37}}\hat{\imath} + \dfrac{6}{\sqrt{37}}\hat{\jmath}$

(d) Sketch the position vector $\vec{r}(t)$ and the tangent vector $\vec{r}'(t)$ for $t=0$.

Ex 7: Find the parametric equation for the tangent line to the helix $x = 2\cos t$, $y = 2\sin t$, $z = t$ at the point $(0, 2, \pi/2)$.

► This point corresponds to $t = \pi/2$

$\vec{r}'(t) = \langle -2\sin t, 2\cos t, 1 \rangle \Rightarrow \vec{r}'(\pi/2) = \langle -2, 0, 1 \rangle$

Thus: tangent line is given by $\boxed{x = -2s,\ y = 2,\ z = \pi/2 + s,\ s \in \mathbb{R}}$

Integrals of vector functions

Integrals are also defined component-wise, e.g. for $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\boxed{\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle}$$

Ex 8: Evaluate $\int_0^1 (te^{t^2}\hat{\imath} + 2\sin t\cos t\,\hat{\jmath} + e^{5t}\hat{k})\,dt$.

► $\int_0^1 te^{t^2}dt = \dfrac{e^{t^2}}{2}\Big|_{t=0}^{t=1} = \dfrac{e-1}{2}$; $\int_0^1 2\sin t\cos t\,dt = \sin^2 t\Big|_{t=0}^{t=1} = \sin^2(1)$; $\int_0^1 e^{5t}dt = \dfrac{e^{5t}}{5}\Big|_{t=0}^{t=1} = \dfrac{e^5-1}{5}$

So: get $\boxed{\dfrac{e-1}{2}\hat{\imath} + \sin^2(1)\hat{\jmath} + \dfrac{e^5-1}{5}\hat{k}}$

* Length of a curve

For the case of plane curves given by $x = f(t)$, $y = g(t)$, many of you know that the length of the curve for $a \le t \le b$ equals

$$\boxed{L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt}$$

Likewise, given a space curve $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$, its length equals

$$\boxed{L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt}$$

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Both formulas for L may be uniformly written as

$$\boxed{L = \int_a^b \|\vec{r}\,'(t)\|\, dt}$$

Recalling $\|\vec{r}\,'(t)\| = $ speed, this formula becomes obvious

! We should request above that the curve is traversed exactly once.

Ex 9: (a) Find the length of the curve given by

$\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ from $(3,0,0)$ to $(3,0,16\pi)$

$$L = \int_0^{4\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2}\, dt = \int_0^{4\pi} 5\, dt = 20\pi$$

(b) —//— $\vec{r}(t) = \langle 6t, t^3, 3t^2\rangle$, $-1 \le t \le 1$.

$$L = \int_{-1}^{1} \sqrt{6^2 + (3t^2)^2 + (6t)^2}\, dt = \int_{-1}^{1} \sqrt{9t^4 + 36t^2 + 36}\, dt = \int_{-1}^{1} (3t^2+6)\, dt = (t^3+6t)\Big|_{-1}^{1} = 14$$

* Velocity & Acceleration

Imagine a particle moving in $\mathbb{R}^3$ with position vector $\vec{r}(t)$. Then, the velocity is the limit of the displacement vector and equals

$$\boxed{\vec{v}(t) = \vec{r}\,'(t)}$$

The speed at time t equals $\|\vec{v}(t)\| = \|\vec{r}\,'(t)\|$.

The acceleration of the particle is defined as the derivative of the velocity

$$\boxed{\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)}$$

Ex 10: A moving particle stands at the initial position $\vec{r}(0) = \langle 2,0,3\rangle$, with initial velocity $\vec{v}(0) = \langle 1,2,-3\rangle$ and acceleration $\vec{a}(t) = \langle e^t, -2t, 2\sin t\rangle$. Find its velocity & position at time t.

$$\vec{a}(t) = \vec{v}\,'(t) \Rightarrow \vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\, du = \langle 1,2,-3\rangle + \int_0^t \langle e^u, -2u, 2\sin u\rangle\, du$$

$$= \langle 1,2,-3\rangle + \langle e^t - 1, -t^2, -2\cos t + 2\rangle = \boxed{\langle e^t, 2-t^2, -1-2\cos t\rangle}$$

$$\vec{v}(t) = \vec{r}\,'(t) \Rightarrow \vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\, du = \langle 2,0,3\rangle + \int_0^t \langle e^u, 2-u^2, -1-2\cos u\rangle\, du$$

$$= \langle 2,0,3\rangle + \langle e^t - 1, 2t - \tfrac{t^3}{3}, -t - 2\sin t\rangle = \boxed{\langle e^t + 1, 2t - t^3/3, 3 - t - 2\sin t\rangle}$$