Lecture #5

* Organizational

- Office hours
- Peer tutors
- hand out homework #1.

* Last time

Discussed:

- vector functions and space curves
- limits, derivatives, integrals of vector functions
- length of curves

* Velocity & Acceleration – see last page from Lecture #4 Notes

* Functions of several variables – main topic for today.

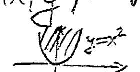
Def: A function of two variables is a function whose domain D is a subset of $\mathbb{R}^2$ and whose range is a subset of $\mathbb{R}$.

We will often write $z = f(x,y)$ (but you can use any other letters!)

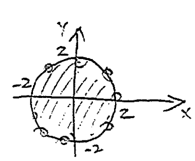
Convention: If a function f is given by the formula and no domain is specified, then the domain of f is the set of all points $(x,y) \in \mathbb{R}^2$ such that $f(x,y)$ is well-defined, while the range of f is the set of all possible values $f(x,y)$.

Ex 1: Sketch the domain for each of the following functions:

(a) $f(x,y) = \sqrt[6]{y - x^2}$

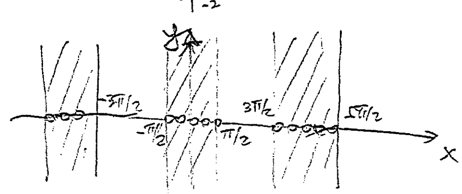


(b) $f(x,y) = \ln(4 - x^2 - y^2)$



boundary excluded!

(c) $f(x,y) = \dfrac{\sqrt{\cos x}}{y}$



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Ex2: Find the range of all 3 functions in Ex1.

(a) $[0, +\infty)$

(b) $(-\infty, \ln 4]$ Provide arguments!

(c) $\mathbb{R} = (-\infty, \infty)$

* Graphs and level curves

The most common way to visualize a function of two variables is by considering the corresponding graph:

Def: If f is a function of two variables with domain D, then the graph of f is the set of all points $(x,y,z) \in \mathbb{R}^3$ s.t. $(x,y) \in D$ & $z = f(x,y)$

Note: This is in a complete analogy by drawing a graph of $y = f(x)$.

An alternative way to visualize $f(x,y)$ is by drawing level curves:

Def: The level curves of a function f of two variables are the curves with equation $f(x,y) = k$, where k is a constant (in the range of f)

Ex 3: Sketch the graphs of the following functions:

(a) $f(x,y) = 1 - x - y$

(b) $f(x,y) = \sqrt{9 - x^2 - y^2}$

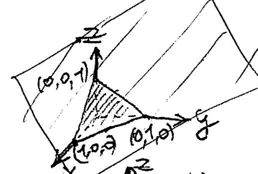
(c) $f(x,y) = x^2 + 9y^2$

(d) $f(x,y) = \sqrt{4 - y^2}$

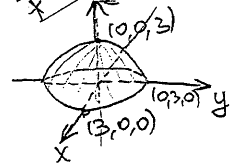
(e) $f(x,y) = 3\sqrt{x^2+y^2}$

(a) $z = 1 - x - y \iff x + y + z = 1$ — plane.

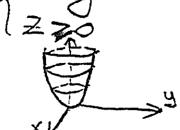
Interceps are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$



(b) $z = \sqrt{9-x^2-y^2} \iff \begin{cases} x^2+y^2+z^2 = 9 \\ z \geq 0 \end{cases}$ — half sphere



(c) $z = x^2 + 9y^2$



elliptic paraboloid

Note that intersecting with xz-plane get parabola $z = x^2$

yz-plane get parabola $z = 9y^2$

any plane $z = k > 0$ get ellipses

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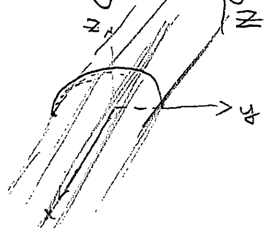
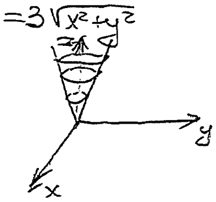
(d) $z = \sqrt{4-y^2} \iff \begin{cases} y^2+z^2=4 \\ z \geq 0 \end{cases}$ ← independent of x.



– half of cylinder

(e) $z = 3\sqrt{x^2+y^2}$

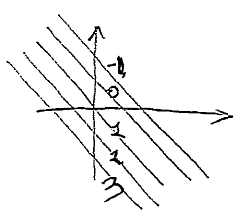


← cone

intersection with xz-plane – 2 halves of lines
yz-plane – 2 halves of lines
plane $z=k>0$ – circles

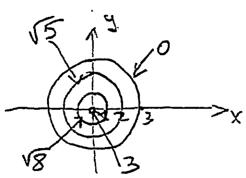
Ex 4: Draw the level curves in each of the above 5 cases.

(a) Level curves are given by solving $k = 1-x-y \iff x+y = 1-k$.



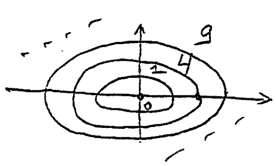
! The numbers over the level curves keep track of the level k.

(b) Level curves: $k = \sqrt{9-x^2-y^2} \iff \begin{cases} x^2+y^2 = 9-k^2 \\ 0 \leq k \end{cases}$



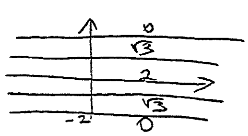
(circles) ← All circles have radius ≤ 3.

(c) Level curves: $k = x^2+9y^2$. Clearly $k \in [0,\infty)$



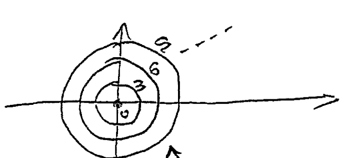
(ellipses)

(d) Level curves: $k = \sqrt{4-y^2} \iff \begin{cases} y = \pm\sqrt{4-k^2} \\ k \geq 0 \end{cases}$



(lines) ← All lines are bounded by $y=2$ on top and $y=-2$ on the bottom.

(e) Level curves: $k = 3\sqrt{x^2+y^2} \iff \begin{cases} x^2+y^2 = \frac{k^2}{9} \\ k \geq 0 \end{cases}$



(circles) but unbounded unlike in (b)

the circles are evenly-spaced, in contrast to e.g. $f(x,y) = x^2+y^2$

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Remarks: (a) Do not forget to put numbers over level curves.

(b) By analogy with topographic maps of mountains, we see that the corresponding graph is steep when the level curves are close enough, while it is flatter if they are farther apart.

Ex5: (a) How should we modify function $f(x,y)$ to flip upside down the corresponding graph?

(b) —"— to make it 3 times steeper?

(c) —"— to move it from the origin to a point $(-1,3)$

▸ (a) $f(x,y) \rightsquigarrow -f(x,y)$

(b) $f(x,y) \rightsquigarrow 3f(x,y)$

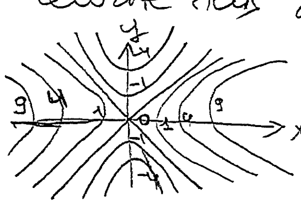
(c) $f(x,y) \rightsquigarrow f(x+1,y-3)$ ▪

Ex6: Draw the level curves of the function $f(x,y)=x^2-y^2$.

▸ Level curves: $k=x^2-y^2$

If $k=0$, this gives two lines $y=\pm x$.

If $k\neq 0$, rewrite this as $k=(x-y)(x+y) \Rightarrow$ get rotated hyperbola.



▪

! The graph of this function is called hyperbolic paraboloid.

The reason for such a name is that intersecting with yz-plane, we obtain parabola $z=-y^2$ (looking downstairs), with xz-plane get parabola $z=x^2$ (looking upstairs), with planes $z=k\neq 0$ get hyperbolas.

Ex7: Hand out matching game!