LECTURE #6 09/17/2019

* Last time
  - Level curves & graphs
    ↳ cover last page of Lecture 5 notes

* Partial derivatives

Given $f=f(x,y)$ and a point (a,b), we define its 1^{st} order partial derivatives

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x}(a,b) &= f_x(a,b) = \lim_{t\to 0} \frac{f(a+t,b)-f(a,b)}{t} \\ \frac{\partial f}{\partial y}(a,b) &= f_y(a,b) = \lim_{t\to 0} \frac{f(a,b+t)-f(a,b)}{t} \end{aligned}}$$

! Same applies to $f=f(x,y,z)$ giving rise to f_x, f_y, f_z

Other way to think about is that we freeze all other coordinates. For example, if we set $g(x) := f(x,b)$, then $\boxed{f_x(a,b) = g'(a)}$. Likewise, if $h(y) := f(a,y)$, then $\boxed{f_y(a,b) = h'(b)}$

Ex1: For $f(x,y,z) = \cos(x^2y) + e^{xy} + \sin(yz)$, find f_x, f_y, f_z.

$$f_x(x,y,z) = -\sin(x^2y)\cdot 2xy + e^{xy}\cdot y$$
$$f_y(x,y,z) = -\sin(x^2y)\cdot x^2 + e^{xy}\cdot x + \cos(yz)\cdot z$$
$$f_z(x,y,z) = \cos(yz)\cdot y$$

Rmk: Geometrically one can think of $f_x(a,b)$ as of the slope of the curve in the plane $y=b$ obtained by intersecting the plane $y=b$ with the graph of function $f(x,y)$. Similar applies to $f_y(a,b)$, but now we intersect the graph with plane $x=a$

* Higher Derivatives

If $f=f(x,y)$, then $f_x(x,y)$ and $f_y(x,y)$ are again functions of 2 variables and we can compute their partials:

$$\boxed{(f_x)_x = f_{xx},\ (f_x)_y = f_{xy},\ (f_y)_x = f_{yx},\ (f_y)_y = f_{yy}} \leftarrow \text{2nd order partial derivatives of } f$$

also denoted $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y^2}$

Ex2: For $f(x,y) = \cos(xe^y)$ compute $f_{xx}, f_{xy}, f_{yx}, f_{yy}$.

LECTURE #6

As observed in Ex2, $f_{xy} = f_{yx}$ which is not just a coincidence.

Theorem (Clairaut's theorem): If f_{xy} and f_{yx} are continuous, then

$$\boxed{f_{xy} = f_{yx}}$$

Consequence: When computing higher order partial derivatives, you may pick the order in the best way for computations, e.g.

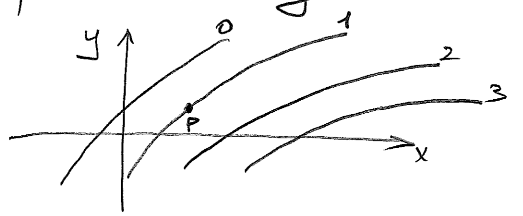
$$f_{xxyyy} = f_{xyyyx} = f_{yxyxy} = \dots$$

Ex3: Verify that $u(x,t) = \sin(x-at)$ satisfies the wave equation $u_{tt} = a^2 u_{xx}$

$$\left.\begin{array}{l} u_t = \cos(x-at)\cdot(-a) \Rightarrow u_{tt} = -\sin(x-at)\cdot(-a)^2 \\ u_x = \cos(x-at) \Rightarrow u_{xx} = -\sin(x-at) \end{array}\right\} \Rightarrow u_{tt} = a^2 u_{xx}$$

$\blacksquare$

Ex 4: Given level curves of $f(x,y)$ below, determine if f_x, f_y are positive or negative at the point P.



$f_x(P) > 0$ as f increases as we move in x-direction from P

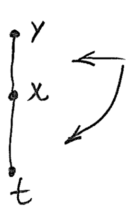
$f_y(P) < 0$ as f decreases as we move in y-direction from P $\blacksquare$

* **Chain Rule**

Let us recall the classical chain rule from high-school:

$$y = f(x),\ x = g(t) \Rightarrow y = f(g(t)) \text{ and } \underline{\frac{d}{dt}y = \frac{dy}{dx}\cdot\frac{dx}{dt} = f'(g(t))\cdot g'(t)}$$

We shall depict this case as follows



meaning top variable is a function of the bottom one

So: Derivative $\frac{dy}{dt}$ arises as a product of consequent derivatives $\frac{dy}{dx}$ and $\frac{dx}{dt}$ as we move from x to t.

LECTURE #6

Now we would like to get the chain rule for the case when functions of several variables are involved.

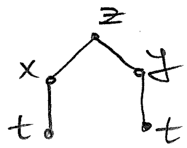
Ex5: If $z = x^2 + e^{3y}$, $x = \sin(t^2)$, $y = \cos(t^3)$, find $\left.\frac{\partial z}{\partial t}\right|_{t=0}$.

One way: Express z explicitly via t and then differentiate:

$$z = \sin^2(t^2) + e^{3\cos(t^3)} \Rightarrow \frac{dz}{dt} = 2\sin(t^2)\cos(t^2)\cdot 2t + e^{3\cos(t^3)}\cdot(-3\sin(t^3))\cdot 3t^2$$

$$\Rightarrow \left.\frac{dz}{dt}\right|_{t=0} = 0$$

Uniform Solution: Start by drawing the dependence of variables



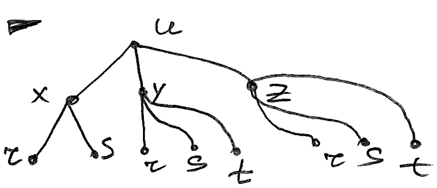
The rule now to add all contributions along each path from top (z) to end-vertex t, where each contribution is a product of consequent partials. Explicitly:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = 2x\cdot\cos(t^2)\cdot 2t + e^{3y}\cdot 3\cdot(-\sin(t^3))\cdot 3t^2.$$

If the question was to evaluate in general, you would still need to express x, y via t in the above formula. But as we need value at $t=0 \Rightarrow x=0, y=1 \Rightarrow \left.\frac{\partial z}{\partial t}\right|_{t=0} = 0+0=0$ ∎

Key Idea: Always start by drawing a picture as above.

Ex6: If $u = x^2y + y^2z^3$, $x = re^s$, $y = \sin(r+s)t^2$, $z = e^{r-s}\cdot\cos(t)$, find $\frac{\partial u}{\partial t}$ at $r=1, s=0, t=0$.



So: $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}$

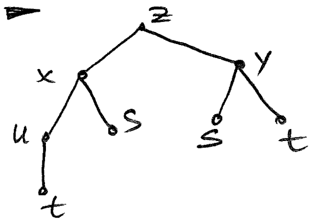
$$= (x^2 + 2yz^3)\cdot\sin(r+s)\cdot 2t + 3y^2z^2\cdot e^{r-s}(-\sin(t))$$

At $r=1, s=0, t=0 \Rightarrow x=1, y=0, z=e$

Thus: $\left.\frac{\partial u}{\partial t}\right|_{r=1,s=0,t=0} = 0+0=0$ ∎

LECTURE #6

Ex7: If $z = e^x \sin(y)$, $x = u^2 s$, $y = s^2 t^3$, $u = e^t$, find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$.



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} =$$

$$= e^x \sin(y)\cdot u^2 + e^x \cos(y)\cdot 2st^3$$

$$= e^{e^{2t}s}\sin(s^2t^3)\cdot e^{2t} + e^{e^{2t}s}\cos(s^2t^3)\cdot 2st^3$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= e^x\sin(y)\cdot 2us\cdot e^t + e^x\cos(y)\cdot 3s^2t^2$$

$$= e^{e^{2t}s}\sin(s^2t^3)\cdot 2e^{2t}s + e^{e^{2t}s}\cos(s^2t^3)\cdot 3s^2t^2$$ ∎

* Implicit differentiation

Suppose y is a function in x, $y = y(x)$, which is not known explicitly, BUT we rather know that $F(x,y) = 0$ for a given function $F(\cdot,\cdot)$.

Want: Compute $\frac{\partial y}{\partial x}$.

As $0 = F(x, y(x))$, differentiating with respect to x, we find

$$0 = \frac{\partial}{\partial x} F(x, y(x)) = F_x \cdot \underbrace{\frac{\partial x}{\partial x}}_{1} + F_y \cdot \frac{\partial y}{\partial x} \Rightarrow \boxed{\frac{\partial y}{\partial x} = -\frac{F_x}{F_y}}$$

Ex8: Find y' given $x^3 + e^{2y} = 2x^2y^2$

This equation is equivalent to $F(x,y) = 0$, where $F(x,y) = x^3 + e^{2y} - 2x^2y^2$.

So: $y' = \frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{3x^2 - 4xy^2}{2e^{2y} - 4x^2y}$ ∎