LECTURE #8 09/24/2019

* Last time

- Directional derivatives
  - If $\vec{u}$ is a unit vector in $\mathbb{R}^2$, $\vec{u} = \langle a, b \rangle$, then

$$\boxed{D_{\vec{u}} f(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0+ta, y_0+ta) - f(x_0, y_0)}{t} \overset{\text{chain rule}}{=} f_x(x_0,y_0)\cdot a + f_y(x_0,y_0)\cdot b = \nabla f(x_0,y_0) \cdot \vec{u}}$$

  - If $\vec{u}$ is a unit vector in $\mathbb{R}^3$, $\vec{u} = \langle a, b, c \rangle$, then

$$\boxed{D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{t \to 0} \frac{f(x_0+ta, y_0+tb, z_0+tc) - f(x_0, y_0, z_0)}{t} = f_x(x_0,y_0,z_0)\cdot a + f_y(x_0,y_0,z_0)\cdot b + f_z(x_0,y_0,z_0)\cdot c = \nabla f(x_0,y_0,z_0)\cdot \vec{u}}$$

  - If u is not a unit vector, replace it by $\hat{u} = \frac{\vec{u}}{\|\vec{u}\|}$ and apply above formula

Note: $D_{\hat{i}} f = f_x$, $D_{\hat{j}} f = f_y$, $D_{\hat{k}} f = f_z$

- Gradient
  - For a function of two variables $f(x,y)$, set

$$\boxed{\nabla f(x_0, y_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0) \rangle}$$

  - For a function of three variables $f(x,y,z)$, set

$$\boxed{\nabla f(x_0,y_0,z_0) = \langle f_x(x_0,y_0,z_0), f_y(x_0,y_0,z_0), f_z(x_0,y_0,z_0) \rangle}$$

- Since $D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u} = \|\nabla f(P)\| \cdot \underbrace{\|\vec{u}\|}_{1} \cdot \cos\theta$, we get: ($\theta$ = angle between $\vec{u}$ and $\nabla f(P)$)

$$\boxed{-\|\nabla f(P)\| \le D_{\vec{u}} f(P) \le \|\nabla f(P)\|}$$

equality holds iff $\vec{u}$ is in the opposite direction to $\nabla f(P)$, i.e. $\vec{u} = -\frac{\nabla f(P)}{\|\nabla f(P)\|}$

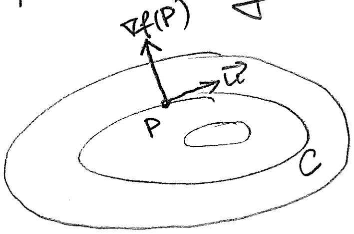
equality holds iff $\vec{u}$ is in the direction of $\nabla f(P)$, i.e. if $\vec{u} = \frac{\nabla f(P)}{\|\nabla f(P)\|}$ (assuming $\vec{u}$ is unit)

- Tangent plane (resp. tangent line) to level surface (resp. level curve) $F = k$ (function level) at point P is defined as passing through P and orthogonal to $\nabla f(P)$

LECTURE #8

- Geometric Meaning of the gradient

Let us illustrate the concept of the gradient for functions of two variables.
Let $f = f(x,y)$, $P(x_0,y_0)$ – any point in the domain. Consider the level curve C of f containing P, i.e. $C = \{(x,y) \mid f(x,y) = f(x_0,y_0)\}$.



Clearly as we move along C, the value of f does not change. In particular, we get $D_{\vec{u}} f(P) = 0$, where $\vec{u}$ is a tangent vector to C at point P.

Thus: $0 = D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u} \Rightarrow$ $\boxed{\nabla f(P) \text{ is orthogonal to the tangent vector } \vec{u}}$

$\Downarrow$

this explains why tangent line to C at P is perpendicular to $\nabla f(P)$

Moral: $\boxed{\text{At any point } P \text{ of any level curve } C \text{ of } f(x,y), \text{ the gradient vector } \nabla f(P) \text{ is perpendicular to } C \text{ at } P}$

Consequence: Tangent line to C at $P(x_0,y_0)$ is $\boxed{f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0}$

Completely similarly, given a level surface S of $f(x,y,z)$ and a point $P(x_0,y_0,z_0)$ on S, we find:

$\boxed{\nabla f(P) \text{ is orthogonal to the tangent vector of any curve } C \text{ lying on } S \text{ and passing through } P \text{ at the point } P}$

Consequence: Tangent plane to S at $P(x_0,y_0,z_0)$ is given by the following equation

$$\boxed{f_x(x_0,y_0,z_0)\cdot(x-x_0) + f_y(x_0,y_0,z_0)\cdot(y-y_0) + f_z(x_0,y_0,z_0)\cdot(z-z_0) = 0}$$

- Key Example: Graph of function of 2 variables

If S is a graph of $f(x,y)$, i.e. $S = \{(x,y,z) \mid z = f(x,y)\}$, then it can be realized as a level surface of the function $F(x,y,z) = f(x,y) - z$, so that

$\boxed{\nabla F(x_0,y_0,z_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0), -1\rangle \Rightarrow \text{Tangent Plane}: f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0) - (z-z_0) = 0}$

Note: We could find the normal vector as a cross-product of tangent vectors $\vec{v_1}$ to $C_1 = \{(x, y_0, f(x,y_0))\}$ and $\vec{v_2}$ to $C_2 = \{(x_0, y, f(x_0,y))\}$.

$\vec{v_1} = \langle 1, 0, f_x\rangle$, $\vec{v_2} = \langle 0, 1, f_y\rangle \Rightarrow \vec{v_1} \times \vec{v_2} = \langle -f_x(x_0,y_0), -f_y(x_0,y_0), 1\rangle = \nabla F(x_0,y_0,z_0)$

LECTURE #8

Ex 1: Find equation of the tangent plane and the normal line to the surface S given by $z = e^{\sin(x+y^2)}$ at the point $P(\pi, 0, 1)$.

$F(x,y,z) = e^{\sin(x+y^2)} - z$

$\nabla F = \langle e^{\sin(x+y^2)} \cdot \cos(x+y^2),\ e^{\sin(x+y^2)} \cdot \cos(x+y^2) \cdot 2y,\ -1 \rangle$

$\Downarrow$

$\nabla F(P) = \langle -1, 0, -1 \rangle$

Tangent plane: $(x-\pi) + (z-1) = 0$

Normal line: $x = \pi - t,\ y = 0,\ z = 1 - t$

* Min/Max Problems (Section 14.7)

Goal: Find max/min values of $f(x,y)$ defined on certain domains $D \subseteq \mathbb{R}^2$.

Let us first recall how similar problems are done for $f(x)$:

Ex 2: Find the (absolute) maximal & minimal value of $f(x) = x^3 - 12x$ on $[-3, 5]$.

- Critical points: $0 = f'(x) = 3x^2 - 12 \Rightarrow x = \pm 2$ and $f(2) = -16$, $f(-2) = 16$
- Also check end-points: $f(-3) = 9$, $f(5) = 65$.

So: Max is 65 (at $x=5$), Min is -16 (at $x=2$)

Def: (a) Function $f(x,y)$ has a local maximum at (a,b) if $f(x,y) \leq f(a,b)$ for any (x,y) "near" (a,b). The number $f(a,b)$ is a local max value.

(b) Function $f(x,y)$ has a local minimum at (a,b) if $f(x,y) \geq f(a,b)$ for any (x,y) "near" (a,b). The number $f(a,b)$ is a local min value.

(c) If the inequalities in (a) or (b) hold for any (x,y) in the domain, then (a,b) is called absolute max or min of $f(x,y)$, while $f(a,b)$ is the the absolute max or min value.

Note that considering functions $g(x) := f(x,b)$, $h(y) = f(a,y)$, we see that

$$\left.\begin{array}{l} x = a \text{ - local max/min of } g(x) \\ y = b \text{ - local max/min of } h(y) \end{array}\right\} \Rightarrow f_x(a,b) = f_y(a,b) = 0$$

Theorem: If $f(x,y)$ has a local max or min at (a,b) and f_x, f_y - exist, then

$$\boxed{f_x(a,b) = f_y(a,b) = 0}$$

Def: A point (a,b) in the domain of f is called critical if $f_x(a,b)=f_y(a,b)=0$ (or one of these partials does not exist).

Warning: Local Max/Min $\Rightarrow$ Critical

BUT

Critical $\not\Rightarrow$ Local Max/Min

Geometric Meaning: As just discussed in the beginning of the class, the equalities $f_x(a,b)=0=f_y(a,b)$ geometrically mean that the tangent plane to the graph of $f(x,y)$ at the point $(a,b,f(a,b))$ is parallel to xy-plane

Ex3: Find local max/min values of $f(x,y)=x^2-y^2$.

$f_x=2x$, $f_y=-2y \Rightarrow$ the only critical point is $(0,0)$.
However, $f(x,0)=x^2>0$ as x approaches 0
$f(0,y)=-y^2<0$ as y approaches 0

So: f has no local max/min values at all.

Theorem (Second Derivative Test): Suppose second partial derivatives of f are continuous near (a,b) and assume $f_x(a,b)=f_y(a,b)=0$. Define

$$\boxed{D:=D(a,b)=f_{xx}(a,b)\,f_{yy}(a,b)-(f_{xy}(a,b))^2}$$

(a) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ – local minimum

(b) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ – local maximum

(c) If $D<0$, then $f(a,b)$ is NOT a local max/min.

(d) If $D=0$, then nothing can be said, i.e. the test is inconclusive

Def: If $D<0$, then (a,b) is called a saddle point of f.

Ex4: Find local max/min/saddle points of $f(x,y)=y\sin(x)$.

$f_x=y\cos x$, $f_y=\sin x \Rightarrow$ Critical points are $\{(\pi k,0)\mid k\text{-integer}\}$.

$D=f_{xx}(\pi k,0)f_{yy}(\pi k,0)-f_{xy}(\pi k,0)^2=-1<0$

So: There are no local max/min values, while saddle points are $\{(\pi k,0)\mid k\text{-integer}\}$

LECTURE #8

Ex 5: Find the local max/min values and saddle points of $f(x,y) = 5 - x^4 + 2x^2 - y^2$.

$\left.\begin{array}{l} f_x(x,y) = -4x^3 + 4x = 4x(1-x)(1+x) \\ f_y(x,y) = -2y \end{array}\right\} \Rightarrow$ Critical points: $(0,0)$, $(-1,0)$, $(1,0)$.

$f_{xx}(x,y) = 4 - 12x^2$, $f_{yy}(x,y) = -2$, $f_{xy}(x,y) = 0 \Rightarrow D = 2(12x^2 - 4)$.

- At point $(0,0)$: $D = -8 < 0 \Rightarrow (0,0)$ - saddle point
- At $(-1,0)$: $D = 16 > 0$, $f_{xx}(-1,0) = -8 < 0 \Rightarrow (-1,0)$ - local max
- At $(1,0)$: $D = 16 > 0$, $f_{xx}(1,0) = -8 < 0 \Rightarrow (1,0)$ - local max

Also: $f(-1,0) = 6 = f(1,0)$

So: There are no local min, local max are at $(\pm 1, 0)$ with values $= 6$, saddle points: $(0,0)$

Ex 6: Find the point on the plane $x + y + 5z - 1 = 0$ that is closest to $P(1,2,5)$

Distance $d = \sqrt{(x-1)^2 + (y-2)^2 + (z-5)^2}$

Want: Minimize d or equivalently d^2 (as $d \geq 0$).

Plane equation $\Rightarrow x = 1 - y - 5z \Rightarrow d^2 = \underbrace{(y-2)^2 + (z-5)^2 + (y+5z)^2}_{f(y,z)}$

$f_y(y,z) = 2(y-2) + 2(y+5z) = 2(2y + 5z - 2)$

$f_z(y,z) = 2(z-5) + 10(y+5z) = 2(5y + 26z - 5)$

Solve $\begin{cases} 2y + 5z - 2 = 0 \\ 10y + 52z - 10 = 0 \end{cases} \Rightarrow \begin{cases} z = 0 \\ y = 1 \end{cases} \Rightarrow x = 1 - 1 - 0 = 0$

Thus: there is only one critical point $(1,0)$.

$f_{yy}(1,0) = 4$, $f_{zz}(1,0) = 52$, $f_{yz}(1,0) = 10 \Rightarrow \left.\begin{array}{l} D = 4 \cdot 52 - 10^2 > 0 \\ f_{yy}(1,0) > 0 \end{array}\right\} \Rightarrow (1,0)$ - local min of f.

So: $(0,1,0)$ is a local minimum, but recalling the geometric interpretation it is clear it must be also the absolute minimum.