Lecture #9

* Last time
  - Critical points for $f(x,y)$: solve $f_x(x,y)=0=f_y(x,y)$
  - Second derivative test
    ↳ evaluate $D = f_{xx} f_{yy} - (f_{xy})^2$ for all critical points.

* Today: Absolute Max/Min.

Let us start by recalling the situation for functions of 1 variable.

Ex 0: Find the absolute max and min values of the function x^3 on
(a) $\mathbb{R}$ ← no max/min
(b) $[-2,2]$ ← -8 & 8

This illustrates the general principle that $f(x)$ may have no absolute max/min on $\mathbb{R}$, but any continuous $f(x)$ achieves absolute max/min on any closed interval $[a,b]$

Ex 1: Find the absolute max and min of x^2-2x+y^2+2y+5 on $\mathbb{R}^2$

▸ $f(x,y) = x^2-2x+y^2+2y+5 = (x-1)^2+(y+1)^2+3$

No absolute max, while absolute minimum $=3$ ▪

As for f-s of 1 variable, there may be no absolute max/min of $f(x,y)$ on $\mathbb{R}^2$. However, if f is continuous on a closed and bounded set $D \subseteq \mathbb{R}^2$, then a general theorem guarantees that absolute max/min are achieved on D!

Rmk: In the last exercice from last time, when computing a distance from a given pt to a given plane, geometry immediately "says" that absolute max is not achieved, abs. min is achieved

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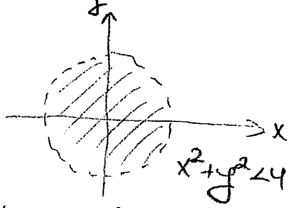
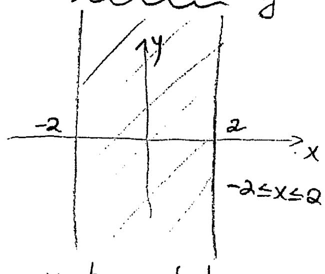
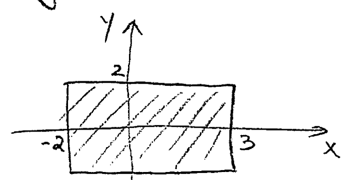
Def: A region $D \subseteq \mathbb{R}^2$ is *bounded* if it is not infinite.



bounded
not closed



unbounded
closed

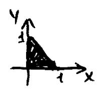


bounded
closed

Def: A region $D \subseteq \mathbb{R}^2$ is *closed* if it includes its boundary

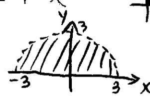
Ex 2: Are the following regions closed or/and bounded:

(a) $0 \le x \le 1,\ 0 \le y \le 1-x$



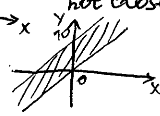
closed & bounded

(b) $0 \le y < \sqrt{9-x^2}$



not closed & bounded

(c) $x \le y \le x+10.$



closed & unbounded

Algorithm to find absolute max/min on the closed & bounded $D \subseteq \mathbb{R}^2$:

* First, find the values of f at the critical points of f inside D
* Second, find the extreme values of f on the boundary of D
* The largest/smallest of the values in the above 2 steps is the absolute max/min of f on D

Warning: When finding extreme values of f on the boundary of D, you CANNOT apply the 2^{nd} Derivative test. Instead, you should split boundary into several pieces, each of which is easy to parametrize by 1 variable, and thus reduce the question to the case of functions of 1 variable

! Note: Do not need to apply 2^{nd} Derivative Test to critical points

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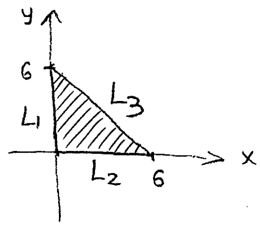
Ex 3: Find the absolute max / min of $f(x,y) = x^2 - 4x + y^2 - 6y + 1$ in the closed triangular region with vertices $(0,0)$, $(6,0)$, $(0,6)$.



Step 1: Critical pts

$$\left.\begin{aligned} f_x(x,y) &= 2x-4 = 0 \text{ iff } x=2 \\ f_y(x,y) &= 2y-6 = 0 \text{ iff } y=3 \end{aligned}\right\} \Rightarrow \text{get only 1 critical pt } (2,3) \leftarrow \text{it is indeed in our region}$$

and $f(2,3) = \boxed{-12}$

Step 2: $\boxed{L_1 = \{(0,y) : 0 \le y \le 6\}}$

$g(y) = f(0,y) = y^2 - 6y + 1$

$g'(y) = 2y - 6 = 0$ iff $y = 3$ and $g(y) = \boxed{-8}$

Also compute at end-points: $g(0) = \boxed{1}$, $g(6) = \boxed{1}$

$\boxed{L_2 = \{(x,0) : 0 \le x \le 6\}}$

$g(x) = f(x,0) = x^2 - 4x + 1$

$g'(x) = 2x - 4 = 0$ iff $x = 2$: $g(2) = \boxed{-3}$

Also end-points: $g(0) = 1$, $g(6) = \boxed{13}$
↑ already listed above

$\boxed{L_3 = \{(x, 6-x) : 0 \le x \le 6\}}$

$g(x) = x^2 - 4x + (36 - 12x + x^2) - (36 - 6x) + 1 = 2x^2 - 10x + 1$

$g'(x) = 4x - 10 = 0$ iff $x = \frac{5}{2}$ and $g(\frac{5}{2}) = \frac{25}{2} - 25 + 1 = \boxed{-\frac{23}{2}}$

Step 3: Choose max & min value of the above listed:

Absolute max $= \underline{13}$ and it is achieved at $\underline{(6,0)}$

Absolute min $= \underline{-12}$ and it is achieved at $\underline{(2,3)}$

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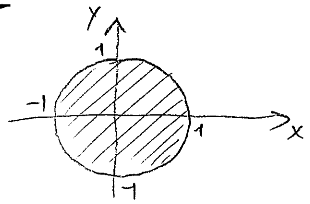
Ex 4: Find the absolute max / min of $f(x,y) = 2x^4 + y^4$ on $D: x^2+y^2 \le 1$.



Critical pts: $f_x = 8x^3 = 0$ iff $x=0$, $f_y = 4y^3 = 0$ iff $y=0$ $\Rightarrow$ critical points: $(0,0)$ and $f(0,0) = \boxed{0}$

Boundary: unit circle, which can be parametrized by $\{(\cos\theta, \sin\theta) \mid 0 \le \theta \le 2\pi\}$.

$g(\theta) = f(\cos\theta, \sin\theta) = 2\cos^4\theta + \sin^4\theta$

$g'(\theta) = -8\cos^3\theta\sin\theta + 4\sin^3\theta\cos\theta = 4\cos\theta\sin\theta(\sin^2\theta - 2\cos^2\theta) = 4\cos\theta\sin\theta(3\sin^2\theta - 2)$

So: $g'(\theta) = 0$ iff $\underbrace{\cos\theta = 0}_{\text{points } (0,1),(0,-1)}$, or $\underbrace{\sin\theta = 0}_{\text{points } (1,0),(-1,0)}$, or $\sin^2\theta = \frac{2}{3} \Rightarrow \cos^2\theta = \frac{1}{3}$

$f(0,1) = f(0,-1) = \boxed{1}$, $f(1,0) = f(-1,0) = \boxed{2}$

Finally, at points $(\cos\theta, \sin\theta)$ such that $\cos^2\theta = \frac{1}{3}$, $\sin^2\theta = \frac{2}{3}$, we have
$$f(\cos\theta, \sin\theta) = 2\cdot\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \boxed{\frac{6}{9}}$$

So: Absolute max $= 2$ and it is achieved at $(\pm 1, 0)$

Absolute min $= 0$ and it is achieved at $(0,0)$ ∎

Ex 5: Find the global (= absolute) max and min of $f(x,y) = x^2y - 2xy + y^2$ on the region $0 \le y \le x$, $0 \le x \le 2$.

Ex 6: Find the global max and min of $f(x,y) = x^2 - y$ on the square $-1 \le x \le 1$, $-1 \le y \le 1$

Ex 7: Find the absolute max and min of $f(x,y) = xy$ on $D: x^2+y^2 \le 1$.

↓ See next pages for solutions

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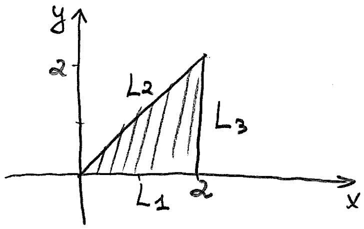
Solution of Ex 5



$$f(x,y) = x^2y - 2xy + y^2$$

Critical pts: $f_x = 2xy - 2y = 2y(x-1) = 0$ iff $y=0$ or $x=1$

$f_y = 2y - 2x + x^2$

So: If $y=0 \Rightarrow x^2-2x=0 \Rightarrow x=0,2 \Rightarrow (0,0), (2,0)$

If $x=1 \Rightarrow 2y-2+1=0 \Rightarrow y=\frac{1}{2} \Rightarrow (1,\frac{1}{2})$ ← Critical pts

$f(0,0)=\boxed{0}$, $f(2,0)=\boxed{0}$, $f(1,\frac{1}{2})=\boxed{-\frac{1}{4}}$

Boundary

- $\boxed{L_1: \{(x,0) \mid 0 \le x \le 2\}}$ $g(x)=f(x,0)=\boxed{0}$ ← constant
- $\boxed{L_2: \{(x,x) \mid 0 \le x \le 2\}}$ $g(x) = f(x,x) = x^3 - 2x^2 + x^2 = x^3 - x^2$

  $g'(x) = 3x^2 - 2x = 3x(x-\frac{2}{3}) \Rightarrow$ crit. pts: $x=0$ ((0,0)–already counted), $x=\frac{2}{3}$ ← $g(\frac{2}{3}) = \frac{8}{27} - \frac{4}{9} = \boxed{-\frac{4}{27}}$

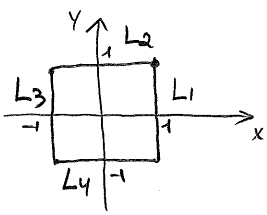
  End points: $g(2) = 8-4 = \boxed{4}$
- $\boxed{L_3: \{(2,y) \mid 0 \le y \le 2\}}$ $g(y) = 4y - 4y + y^2 = y^2$

  $g'(y) = 2y \Rightarrow$ no critical pts inside, while $g(0)=\boxed{0}$, $g(2)=\boxed{4}$

THUS: Absolute max $= 4$, and it is achieved at $(2,2)$

Absolute min $= -1/4$, and it is achieved at $(1,\frac{1}{2})$ ■

Solution of Ex6



$$f(x,y) = x^2 - y$$

Critical pts: $f_x = 2x$, $f_y = -1 \Rightarrow$ NO crit. pts

$\boxed{L_1: \{(1,y) \mid -1 \le y \le 1\}}$ $g(y) = f(1,y) = 1-y$ → max at $y=-1$: $g(-1)=\boxed{2}$; min at $y=1$: $g(1)=\boxed{0}$

$\boxed{L_3: \{(-1,y) \mid -1 \le y \le 1\}}$ $g(y) = f(-1,y) = 1-y$ → max at $y=-1$: $g(-1)=\boxed{2}$; min at $y=1$: $g(1)=\boxed{0}$

$\boxed{L_2: \{(x,1) \mid -1 \le x \le 1\}}$ $g(x) = f(x,1) = x^2-1$ → max at $x=\pm1$: $g(\pm1)=\boxed{0}$; min at $x=0$: $g(0)=\boxed{-1}$

$\boxed{L_4: \{(x,-1) \mid -1 \le x \le 1\}}$ $g(x) = f(x,-1) = x^2+1$ → max at $x=\pm1$: $g(\pm1)=\boxed{2}$; min at $x=0$: $g(0)=\boxed{1}$

THUS: Absolute max $= 2$, and it is achieved at $(\pm1,-1)$

Absolute min $= -1$, and it is achieved at $(0,1)$ ■

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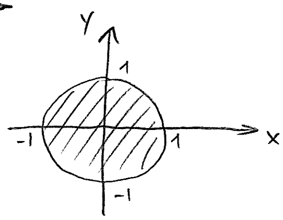
Solution of Ex 7



$f(x,y) = xy$

Critical pts: $\left.\begin{array}{l} f_x = y \\ f_y = x \end{array}\right\} \Rightarrow$ the only critical point is $(0,0)$

$f(0,0) = \boxed{0}$

Boundary: $\boxed{\{(\cos\theta, \sin\theta) \mid 0 \le \theta \le 2\pi\}}$

$g(\theta) = f(\cos\theta, \sin\theta) = \cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$

- max $= \frac{1}{2}$ when $\sin 2\theta = 1$
- min $= -\frac{1}{2}$ when $\sin 2\theta = -1$.

$\sin 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{2}$ or $\frac{5\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$ or $\frac{5\pi}{4} \Rightarrow$ points: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ or $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$\sin 2\theta = -1 \Rightarrow 2\theta = \frac{3\pi}{2}$ or $\frac{7\pi}{2} \Rightarrow \theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4} \Rightarrow$ points $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ or $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

THUS: Absolute max $= \frac{1}{2}$, achieved at $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

Absolute min $= -\frac{1}{2}$, achieved at $\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right)$