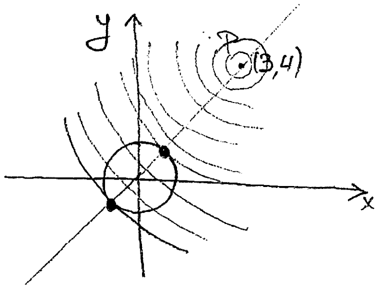
Lecture #10

* Today: Lagrange Multipliers

Ex1: Find a point on the unit circle $x^2+y^2=1$ which is most closely located to the point $P(3,4)$.



Geometric observation: Intuitively it is clear how to find such a point as it suffices to look at the intersections of the circle and the line connecting the center of the circle and P.

However, we want a rigorous approach.

Goal: Minimize $f(x,y)=(x-3)^2+(y-4)^2$ with a constraint $\underbrace{g(x,y)}_{x^2+y^2}=1$

Key Observation: At the points where $f(x,y)$ is min/max, the tangent line to the level curve of f coincides with the tangent line of the curve given by $g(x,y)=1$.

The latter implies that ∇f and ∇g are proportional at the given point. $\Rightarrow$ $\boxed{\nabla f(x,y)=\lambda\cdot\nabla g(x,y)}$

$$\left.\begin{array}{l}\nabla f(x,y)=\langle 2(x-3),\,2(y-4)\rangle\\ \nabla g(x,y)=\langle 2x,2y\rangle\end{array}\right\}\Rightarrow\begin{cases}2(x-3)=\lambda\cdot 2x\\ 2(y-4)=\lambda\cdot 2y\end{cases}\Rightarrow\begin{cases}(1-\lambda)x=3\\ (1-\lambda)y=4\end{cases}\Rightarrow\left.\begin{cases}x=\frac{3}{1-\lambda}\\ y=\frac{4}{1-\lambda}\end{cases}\right\}\Rightarrow$$

But: we also have constraint $x^2+y^2=1$

$$\Rightarrow\left(\frac{3}{1-\lambda}\right)^2+\left(\frac{4}{1-\lambda}\right)^2=1\Rightarrow(1-\lambda)^2=25\Rightarrow\left[\begin{array}{l}\lambda=-4\\ \lambda=6\end{array}\right.$$

If $\lambda=-4\Rightarrow x=\frac{3}{5},\ y=\frac{4}{5}\Rightarrow$ get point $\boxed{\left(\frac{3}{5},\frac{4}{5}\right)}$

If $\lambda=6\Rightarrow x=-\frac{3}{5},\ y=-\frac{4}{5}\Rightarrow$ get point $\boxed{\left(-\frac{3}{5},-\frac{4}{5}\right)}$

Clearly $f\left(\frac{3}{5},\frac{4}{5}\right)<f\left(-\frac{3}{5},-\frac{4}{5}\right)$

$\Rightarrow$ $\left(\frac{3}{5},\frac{4}{5}\right)$ - closest while $\left(-\frac{3}{5},-\frac{4}{5}\right)$ - the most distant

Answer: $\left(\frac{3}{5},\frac{4}{5}\right)$

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The situation is the same for the functions of 3 variables. In other words, if we want to minimize / maximize $f(x,y,z)$ given a constraint $g(x,y,z)=k$, we again solve the following system

$$\begin{cases} \nabla f(x,y,z) = \lambda \cdot \nabla g(x,y,z) \\ g(x,y,z) = k \end{cases} \iff \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = k \end{cases}$$

Def: λ is called a Lagrange multiplier.

Method: ① Find all values (x,y,z) such that there exists λ: $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and also $g(x,y,z)=k$.

② Among the values of f at the points obtained in ① choose the min / max – these are the min / max values of f.

! Rmks: (1) We assume $\nabla g \neq 0$ on the surface $g(x,y,z)=k$.
(2) We assume absolute min / max exist
(3) We don't actually need explicit values for λ.
(4) If f, g – functions of two variables – apply the same procedure.

Ex 2: Find the min / max values of $f(x,y,z) = xyz$ on the sphere $\underbrace{x^2+y^2+z^2}_{g(x,y,z)}=12$

► $\nabla f(x,y,z) = \langle yz, xz, xy \rangle$, $\nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle$

$$\begin{cases} yz = \lambda \cdot 2x & \cdot x \\ xz = \lambda \cdot 2y & \cdot y \\ xy = \lambda \cdot 2z & \cdot z \\ x^2+y^2+z^2 = 12 \end{cases} \Rightarrow xyz = 2\lambda \cdot x^2 = 2\lambda y^2 = 2\lambda z^2.$$

Case 1: $\lambda = 0 \Rightarrow yz = xz = xy = 0 \Rightarrow$ two of the three variables $\{x,y,z\}$ must be 0, while the third is forced to be $\pm\sqrt{12}$ by the condition $x^2+y^2+z^2=12$. At all these points: $\boxed{f=0}$

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Case 2: $\lambda \neq 0 \Rightarrow x^2 = y^2 = z^2$

Combining with constraint $x^2+y^2+z^2=12$ $\Bigg\} \Rightarrow x^2=y^2=z^2=4 \Rightarrow$ 8 points: $(\pm 2, \pm 2, \pm 2)$

At $(2,2,2), (2,-2,-2), (-2,2,-2), (-2,-2,2)$: $f = \boxed{8}$

$(2,2,-2), (2,-2,2), (-2,2,2), (-2,-2,-2)$: $f = \boxed{-8}$

Answer: The max of f is equal to $\underline{8}$ achieved at

$(2,2,2), (2,-2,-2), (-2,2,-2), (-2,-2,2)$

The min of f is equal to $\underline{-8}$ achieved at

$(-2,-2,-2), (2,2,-2), (2,-2,2), (-2,2,2)$.

Ex 3: Find the extreme values of $f(x,y) = e^{xy}$ on the region $D \subseteq \mathbb{R}^2$ given by $D = \{(x,y) \mid x^2+4y^2 \leq 4\}$.

! This problem requires a combination of last lecture & today's lecture

Step 1 (Critical points): $\left.\begin{array}{l} f_x = ye^{xy} = 0 \text{ iff } y=0 \\ f_y = xe^{xy} = 0 \text{ iff } x=0 \end{array}\right\}$ Crit. points: $(0,0) \Rightarrow f(0,0) = \boxed{1}$

Step 2 (Boundary: $\underbrace{x^2+4y^2}_{g(x,y)}=4$)

$\nabla f(x,y) = \langle ye^{xy}, xe^{xy} \rangle$, $\nabla g(x,y) = \langle 2x, 8y \rangle$

$\begin{cases} ye^{xy} = \lambda \cdot 2x & \cdot x \\ xe^{xy} = \lambda \cdot 8y & \cdot y \\ x^2+4y^2=4 \end{cases} \Rightarrow \underbrace{\lambda \cdot 2x^2 = \lambda \cdot 8y^2}= xye^{xy}$

either $\lambda = 0 \Rightarrow x=0, y=0$ which is impossible as $x^2+4y^2=4$

or

$\lambda \neq 0 \Rightarrow 2x^2 = 8y^2 \Rightarrow x^2 = 4y^2 \Big\} \Rightarrow x^2 = 4y^2 = 2$.

But combining this with $x^2+4y^2=4$

$\Rightarrow$ get 4 points $(\sqrt{2}, \frac{1}{\sqrt{2}}), (-\sqrt{2}, -\frac{1}{\sqrt{2}}), (\sqrt{2}, -\frac{1}{\sqrt{2}}), (-\sqrt{2}, \frac{1}{\sqrt{2}})$ with values of f: $\boxed{e, e, \frac{1}{e}, \frac{1}{e}}$

As $\frac{1}{e} < 1 < e \Rightarrow$ Max. of f is $\underline{e}$ achieved at $(\sqrt{2}, 1/\sqrt{2}), (-\sqrt{2}, -1/\sqrt{2})$

Min of f is $\underline{e^{-1}}$ achieved at $(\sqrt{2}, -1/\sqrt{2}), (-\sqrt{2}, 1/\sqrt{2})$ ▥

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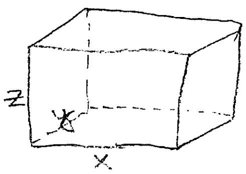
Ex 4: A box is to be constructed with a volume of 500 cubic inches. The box has 4 sides and a bottom, but no top. What are the dimensions of the cheapest box?



$$\text{Volume} = \underbrace{xyz}_{g(x,y,z)} = 500 \text{ inches}^3$$

$$\text{Surface Area} = \underbrace{xy + 2xz + 2yz}_{f(x,y,z)}$$

$\nabla f(x,y,z) = \langle y+2z, x+2z, 2x+2y \rangle$

$\nabla g(x,y,z) = \langle yz, xz, xy \rangle$

$$\left.\begin{cases} y+2z = \lambda \cdot yz & \cdot x \\ x+2z = \lambda \cdot xz & \cdot y \\ 2x+2y = \lambda \cdot xy & \cdot z \\ xyz = 500 \end{cases}\right\} \Rightarrow \lambda xyz = xy+2xz = xy+2yz = 2xz+2yz$$

- $xy+2xz = xy+2yz \Rightarrow z(x-y)=0 \Longrightarrow x=y$ as $z \neq 0$ due to $xyz=500$
- $xy+2yz = 2xz+2yz \Rightarrow x(y-2z)=0 \Rightarrow y=2z$ as $x\neq 0$ —//—

So: $x=y=2z$.

Plugging this into $xyz=500 \Rightarrow 2z \cdot 2z \cdot z = 500 \Rightarrow z=5 \Bigg\} \Rightarrow \begin{cases} x=10 \\ y=10 \\ z=5 \end{cases}$

As there is only 1 candidate $(x=10, y=10, z=5)$, it must be the solution! ∎

Ex 5: Find min/max values of $\ln(x^2+1)+\ln(y^2+1)+\ln(z^2+1)$ given $x^2+y^2+z^2=12$.

Ex 6*: (a) Maximize $x_1 + \_\_ + x_n$ subject to $x_1^2 + \_\_ + x_n^2 = 1$

(b) Maximize $\sqrt[n]{x_1 \cdot \_\_ \cdot x_n}$ subject to $\begin{cases} x_1 + \_\_ + x_n = 1 \\ x_1 \geq 0, \_\_, x_n \geq 0 \end{cases}$