Lecture #11

* Last time: Method of Lagrange Multipliers
  ↳ explain rigorously why we are solving $\nabla f = \lambda \cdot \nabla g$ (see page 972 of textbook)

* Today: Double integrals (Sections 15.1, 15.2)

Goal: Extend the familiar notion of $\int_a^b f(x)dx$ to the case of functions of two variables: $f(x) \rightsquigarrow f(x,y)$.

Warning: We are skipping the rigorous mathematical definition of double integrals - see pages 988–992 of your textbook.

Geometric Meaning: Double integrals can be utilized to compute "oriented" volumes under $z = f(x,y)$ in the same way usual integrals are used to compute "oriented" areas.

Iterated Integrals

Given a function $f(x,y)$ on the rectangle $[a,b]\times[c,d]$, we can evaluate two iterated integrals

$$\boxed{\int_a^b \Big(\underbrace{\int_c^d f(x,y)dy}_{\text{here } x\text{-fixed and we integrate with respect to } y}\Big)dx} \quad \text{and} \quad \boxed{\int_c^d \Big(\underbrace{\int_a^b f(x,y)dx}_{\text{here } y\text{-fixed and we integrate with respect to } x\text{.}}\Big)dy}$$

Ex1: Evaluate the following iterated integrals:

(a) $\int_0^1 \int_0^2 xe^y dy\, dx$ $\left[= \int_0^1 \left(xe^y\Big|_{y=0}^{y=2}\right)dx = (e^2-1)\int_0^1 x\,dx = \frac{e^2-1}{2}\right]$

(b) $\int_0^2 \int_0^1 xe^y dx\, dy$ $\left[= \int_0^2 e^y \cdot \frac{1}{2} dy = \frac{e^2-1}{2}\right]$

Note: The answers to (a) and (b) coincide!

This is not a coincidence as the following Theorem says.

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Give a geometric justification for the Fubini's Theorem

Theorem (Fubini's Theorem): If $f(x,y)$ is continuous on the rectangle $R=[a,b]\times[c,d]=\{(x,y) \mid a\le x\le b \text{ and } c\le y\le d\}$, then

$$\boxed{\iint_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy}$$

In particular, if $f(x,y)=g(x)\cdot h(y)$ on $R=[a,b]\times[c,d]$, then:

$$\boxed{\iint_R g(x)h(y)\,dx\,dy=\int_a^b g(x)\,dx\cdot\int_c^d h(y)\,dy}$$

Important Particular Case

Ex2: Evaluate $\iint_R 2ye^{xy}\,dA$, where $R=[0,1]\times[-1,1]$

$$\iint_R 2ye^{xy}\,dA=\int_{-1}^1\int_0^1 2ye^{xy}\,dx\,dy=\int_{-1}^1\left(2e^{xy}\Big|_{x=0}^{x=1}\right)dy=\int_{-1}^1 2(e^y-1)\,dy=\boxed{2e-\frac{2}{e}-4}$$

Ex3: Find the volume of the solid S that is bounded by the elliptic paraboloid $4x^2+y^2+z=10$, the planes $x=1$, $y=2$, and the three coordinate planes.

First note that $z=10-4x^2-y^2>0$ over the given region $[0,1]\times[0,2]$ in the xy-plane $\Rightarrow$ "oriented" volume = usual volume!

$$Vol=\int_0^1\int_0^2(10-4x^2-y^2)\,dy\,dx=\int_0^1\left((10-4x^2)\cdot2-\frac{8}{3}\right)dx=\int_0^1\left(\frac{52}{3}-8x^2\right)dx=\frac{52}{3}-\frac{8}{3}=\boxed{\frac{44}{3}}$$

Practice Problems:

- Compute $\int_0^1\int_{-2}^1(y^2+y^3\sin x)\,dx\,dy$ $\left[=\int_0^1\left(y^2x-y^3\cos x\right)_{-2}^{1}dy=\int_0^1[3y^2-y^3(\cos(1)-\cos(-2))]dy\right.$ $\left.=\left(y^3-\frac{y^4}{4}(\cos(1)-\cos(-2))\right)\Big|_{y=0}^{y=1}=1-\frac{1}{4}(\cos(1)-\cos(2))\right]$
- Compute $\iint_{[0,1]\times[0,2]}(ye^{xy}+x\sin(xy))\,dA$

$$\left[\begin{aligned}&\int_0^2\int_0^1 ye^{xy}\,dx\,dy+\int_0^1\int_0^2 x\sin(xy)\,dy\,dx=\int_0^2\left(e^{xy}\Big|_{x=0}^{x=1}\right)dy+\int_0^1\left(-\cos(xy)\Big|_{y=0}^{y=2}\right)dx=\\&=\int_0^2(e^y-1)\,dy-\int_0^1(\cos(2x)-1)\,dx=(e^y-y)\Big|_{y=0}^{y=2}-\left(\frac{1}{2}\sin(2x)-x\right)\Big|_{x=0}^{x=1}=e^2-3-\frac{1}{2}\sin(2)+1\end{aligned}\right]$$

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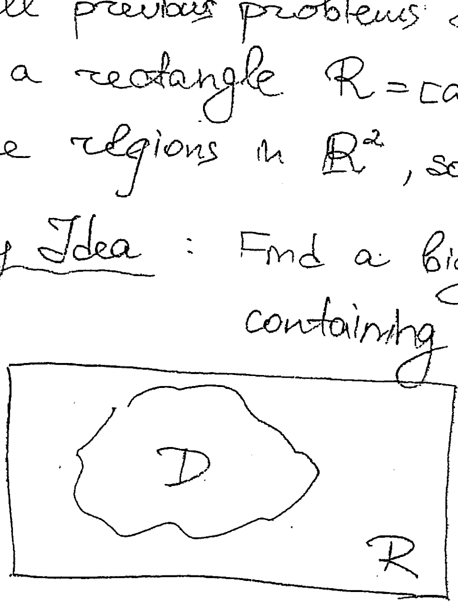
* Double Integrals over general Regions

In all previous problems the region over which we were integrating was a rectangle $R=[a,b]\times[c,d]$. However, there are way more regions in $\mathbb{R}^2$, so we need to integrate over all those.

Key Idea : Find a big enough rectangle $R=[a,b]\times[c,d]$ containing our region D in $\mathbb{R}^2$



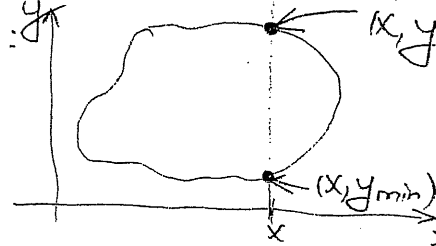
Consider a function $F(x,y)$ on R given by:

$$\boxed{F(x,y)=\begin{cases} f(x,y) & \text{if } (x,y)\in D \\ 0 & \text{if } (x,y)\in R\setminus D \end{cases}}$$

It is clear that the oriented volume under $F(x,y)$ over R is the same as the volume under $f(x,y)$ over D. Hence, we define the double integral of $f(\cdot,\cdot)$ over D via

$$\boxed{\iint_D f(x,y)\,dA=\iint_R F(x,y)\,dA}$$

In practice, this amounts to computing iterated integral, such that the limits of the (inner) integration are no longer fixed constants. E.g. if we use $\iint\ldots dy\,dx$, then the bounds for the inner integral are $\int_{y_{min}}^{y_{max}}\ldots dy$, where:



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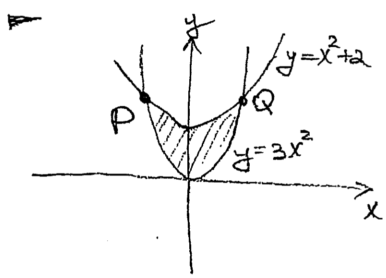
Ex 4: Evaluate $\iint_D (x-y)\,dA$, where D is the region bounded by $y=3x^2$, $y=x^2+2$



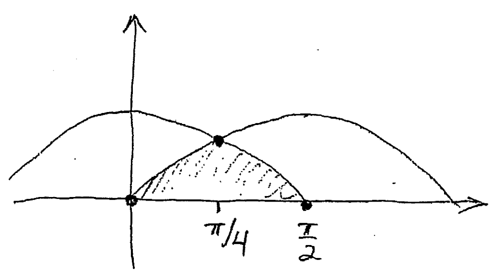
Solve $\underset{x=-1 \text{ or } 1}{\underline{x^2+2=3x^2}}$ to find x-coordinates of the intersection points P, Q.

$$\iint_D (x-y)\,dA = \int_{-1}^{1}\int_{3x^2}^{x^2+2}(x-y)\,dy\,dx = \int_{-1}^{1}\left[x\cdot(2-2x^2)-\frac{y^2}{2}\Big|_{y=3x^2}^{y=x^2+2}\right]dx$$

$$= \int_{-1}^{1}\left(2x-2x^3+\frac{9x^4-x^4-4x^2-4}{2}\right)dx = \int_{-1}^{1}(4x^4-2x^3-2x^2+2x-2)\,dx = \boxed{-\frac{56}{15}}$$

Ex 5: (a) Set up the integral $\iint_D (x-y)\,dA$, where D – bounded by graphs of $y=\sin x$, $y=\cos x$, $0\le x\le\frac{\pi}{2}$, $y\ge 0$

(b) Compute it.



$\sin x=\cos x \Leftrightarrow \tan x=1 \overset{x\in[0,\frac{\pi}{2})}{\Longleftrightarrow} x=\frac{\pi}{4}$

But from the picture we see that we actually have to split this double integral into the sum of two:

$$\underbrace{\boxed{\iint_D (x-y)\,dA = \int_0^{\pi/4}\int_0^{\sin x}(x-y)\,dy\,dx + \int_{\pi/4}^{\pi/2}\int_0^{\cos x}(x-y)\,dy\,dx}}_{\text{this way we set up the integral.}}$$

Important observation is that we need to split into 2 terms!

(b) the answer should be $\pi\left(\frac{7}{8}-\frac{1}{2\sqrt{2}}\right)-\sqrt{2}+\frac{1}{4}$

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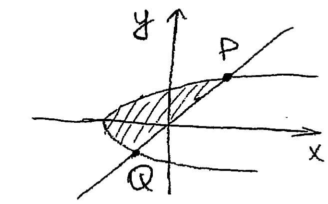
Let us now do a few more interesting problems.

Ex6: Evaluate $\iint_{D} y\,dA$, where D is the region bounded by the line $x=y$ and the parabola $x=y^2-2$.



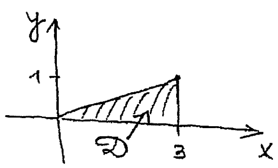
Solve $\underbrace{y^2-2=y}_{y=-1 \text{ or } y=2}$ to find y-coordinates of points P, Q

So: $\iint_{D} y\,dA = \int_{-1}^{2}\int_{y^2-2}^{y} y\,dx\,dy = \int_{-1}^{2} y(y-y^2+2)\,dy = \left(\frac{y^3}{3}-\frac{y^4}{4}+y^2\right)\Big|_{y=-1}^{y=2} = \boxed{\frac{9}{4}}$

Ex7: Evaluate $\int_{0}^{1}\int_{3y}^{3} e^{x^2}\,dx\,dy$

There is no way to compute the inner integral! ↙ Common trick of switching the order of integration

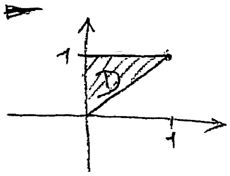
Instead, we shall rewrite it as an iterated integral ending at $dydx$:



$$\int_{0}^{1}\int_{3y}^{3} e^{x^2}\,dx\,dy = \iint_{D} e^{x^2}\,dA = \int_{0}^{3}\int_{0}^{x/3} e^{x^2}\,dy\,dx =$$

$$= \int_{0}^{3} e^{x^2}\cdot\frac{x}{3}\,dx \;\overset{u=x^2}{\underset{du=2x\,dx}{=}}\; \int_{0}^{9} e^{u}\cdot\frac{du}{6} = \boxed{\frac{e^9-1}{6}}$$

Ex8: Evaluate $\iint_{D} \cos(2y^2)\,dA$, where $D=\{(x,y) \mid 0\le y\le 1,\ 0\le x\le y\}$



$$\iint_{D} \cos(2y^2)\,dA = \int_{0}^{1}\int_{0}^{y} \cos(2y^2)\,dx\,dy = \int_{0}^{1} \cos(2y^2)\cdot y\,dy = \frac{\sin(2y^2)}{4}\Big|_{y=0}^{y=1} = \boxed{\frac{\sin(2)}{4}}$$

Warning: If you write rather as $\int_{0}^{1}\int_{x}^{1} \cos(2y^2)\,dy\,dx$ – you will not be able to compute.