LECTURE #12 10/08/2019

* Today: Double Integrals in Polar Coordinates (§15.3 of the textbook)

If we have to integrate $\iint_{\mathcal{D}} f(x,y)\,dA$, where $\mathcal{D}$ is the unit disk $\{(x,y) \mid x^2+y^2 \le 1\}$, then either

$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$ or $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$ are not very convenient due to $\sqrt{\dots}$.

Instead: Use polar coordinates (r,θ) [Recall: $x = r\cos\theta$, $y = r\sin\theta$]

Def: Polar Rectangle is a region in $\mathbb{R}^2$ given in polar coordinates as follows:

$$\boxed{R = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ a \le r \le b\}} \leftarrow \text{Here: } a \ge 0,\ \beta - \alpha \le 2\pi$$

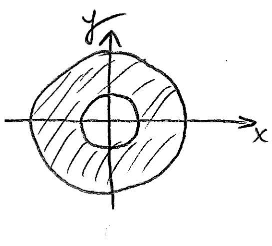
Theorem: If $f(\cdot,\cdot)$ is continuous on the polar rectangle R as above, then

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\cdot \underbrace{r}_{\text{Never forget this extra } r!}\cdot\,dr\,d\theta}$$

Remark: The reason why dA is getting replaced by $r\,dr\,d\theta$ is due to the following picture

Area $\sim dr \cdot r\,d\theta = r\,dr\,d\theta$

Ex 1: Evaluate $\iint_R 4(2x^2 - y^2)\,dA$, where R is bounded by $x^2+y^2=1$, $x^2+y^2=9$



$$\iint_R 4(2x^2-y^2)\,dA = \int_0^{2\pi}\int_1^3 4(2r^2\cos^2\theta - r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} (160\cos^2\theta - 80\sin^2\theta)\,d\theta = \int_0^{2\pi}\left(160\cdot\frac{1+\cos(2\theta)}{2} - 80\cdot\frac{1-\cos(2\theta)}{2}\right)d\theta$$

$$= \boxed{80\pi}$$

For more general regions, we have:

Theorem: If $f(\cdot,\cdot)$ is continuous on the polar region of the form

$$\boxed{D = \{(r,\theta) \mid \alpha \le \theta \le \beta \ \&\ h_1(\theta) \le r \le h_2(\theta)\}}$$

then:

$$\boxed{\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta}$$

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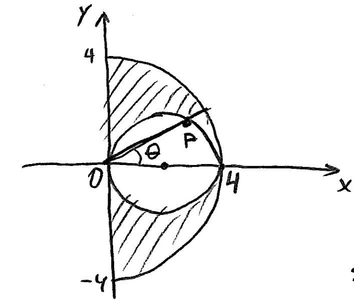
Ex 2: Compute $\iint_D y\,dA$, where D is the region bounded by $x^2+y^2=16$, $x^2+y^2=4x$ in the half-plane $x\geq 0$.



Note: $x^2+y^2=4x \iff (x-2)^2+y^2=4$

To find the distance from O to P, i.e. coordinate r of P, we solve
$r^2=4r\cos\theta \underset{r>0}{\Longrightarrow} r=4\cos\theta$ (as $r=0$ gives the origin)

So: $\iint_D y\,dA = \int_{-\pi/2}^{\pi/2}\int_{4\cos\theta}^{4} r\sin\theta\cdot r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\sin\theta\left(\frac{64}{3}-\frac{64}{3}\cos^3\theta\right)d\theta$

But: $\int_{-\pi/2}^{\pi/2}\sin\theta\,d\theta = -\cos\theta\Big|_{\theta=-\pi/2}^{\theta=\pi/2}=0$

$\int_{-\pi/2}^{\pi/2}\sin\theta\cdot\cos^3\theta\,d\theta \;\overset{u=\cos\theta}{\underset{du=-\sin\theta\cdot d\theta}{=\!=\!=}}\; \int_0^0 u^3\cdot(-du)=0$

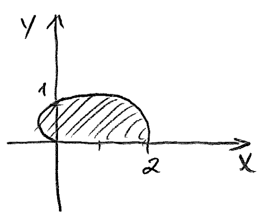
$\Rightarrow \boxed{\iint_D y\,dA=0}$

Remark: The fact we got zero is not surprising as our region D is symmetric w.r.t. reflection in x-axis, while $f(x,y)=y$ is skew-symmetric.

Ex 3: Find the volume of the solid bounded by $z=0$ and $z=4-x^2-y^2$.

$Vol = \iint_{D=\{(x,y):\,4-x^2-y^2\geq 0\}}(4-x^2-y^2)\,dA = \int_0^{2\pi}\int_0^2(4-r^2)r\,dr\,d\theta = \int_0^{2\pi}\left(2r^2-\frac{r^4}{4}\right)\Big|_{r=0}^{r=2}d\theta = \boxed{8\pi}$

Ex 4: Evaluate the integral $\iint_D y\,dA$, where D consists of all points above x-axis and inside the cardioid $r=1+\cos(\theta)$



$\iint_D y\,dA = \int_0^{\pi}\int_0^{1+\cos\theta} r\sin\theta\cdot r\,dr\,d\theta = \int_0^{\pi}\sin\theta\cdot\frac{(1+\cos\theta)^3}{3}\,d\theta$

$\overset{u=1+\cos\theta}{\underset{du=-\sin\theta d\theta}{=\!=\!=}} \int_2^0\frac{u^3}{3}(-du) = \frac{1}{3}\int_0^2 u^3\,du = \frac{16}{12} = \frac{4}{3}$