* Last time: Double integrals in polar coordinates

Ex 1: Find the volume of the solid bounded by two paraboloids $z = 4 - x^2 - y^2$ & $z = 2x^2 + 2y^2 - 2$

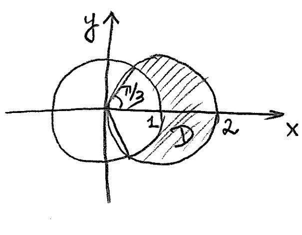
▸ Intersection is a circle, whose projection onto xy-plane consists of (x,y) such that $4 - x^2 - y^2 = 2x^2 + 2y^2 - 2 \Rightarrow x^2 + y^2 = 2$

So: $Vol = \iint_{D: x^2+y^2 \le 2} |(4-x^2-y^2) - (2x^2+2y^2-2)|\, dA = \iint_D (6 - 3x^2 - 3y^2)\, dA = \int_0^{2\pi} \int_0^{\sqrt{2}} (6 - 3r^2) \cdot r\, dr\, d\theta$

$$= \int_0^{2\pi} \left(3r^2 - \frac{3}{4} r^4\right)\Big|_{r=0}^{r=\sqrt{2}} d\theta = 2\pi \cdot (3 \cdot 2 - 3) = \boxed{6\pi}$$

Ex 2: Find the area of the region inside the circle $(x-1)^2 + y^2 = 1$ and outside the circle $x^2 + y^2 = 1$

▸



$$\underset{(D)}{Area} = \iint_D 1\, dA = \int_{-\pi/3}^{\pi/3} \int_1^{?} r\, dr\, d\theta$$

To find "?", solve $(r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1 \Rightarrow r^2 - 2r\cos\theta = 0 \Rightarrow r = 0$ or $r = 2\cos\theta$

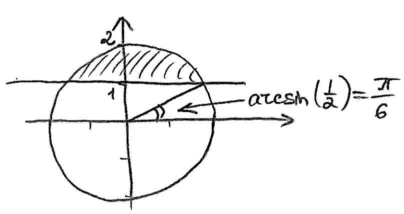
$\Rightarrow$ "?" $= 2\cos\theta$

So: $Area(D) = \int_{-\pi/3}^{\pi/3} \int_1^{2\cos\theta} r\, dr\, d\theta = \int_{-\pi/3}^{\pi/3} \left(\frac{r^2}{2}\right)\Big|_{r=1}^{r=2\cos\theta} d\theta = \int_{-\pi/3}^{\pi/3} \left(2\cos^2\theta - \frac{1}{2}\right) d\theta =$

$$= \int_{-\pi/3}^{\pi/3} \left(\frac{1}{2} + \cos(2\theta)\right) d\theta = \left(\frac{\theta}{2} + \frac{\sin(2\theta)}{2}\right)\Big|_{-\pi/3}^{\pi/3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

Ex 3: Compute $\iint_D x\, dA$, where D is the region inside $x^2 + y^2 = 4$, but above $y = 1$

▸



So: $\iint_D x\, dA = \int_{\pi/6}^{5\pi/6} \int_{1/\sin\theta}^{2} r\cos\theta \cdot r\, dr\, d\theta = \int_{\pi/6}^{5\pi/6} \frac{\cos\theta}{3} r^3 \Big|_{r=\frac{1}{\sin\theta}}^{r=2} d\theta$

$$= \int_{\pi/6}^{5\pi/6} \left(\frac{8}{3}\cos\theta + \frac{1}{3} \frac{\cos\theta}{\sin^3\theta}\right) d\theta = \left(\frac{8}{3}\sin\theta\right)\Big|_{\theta=\pi/6}^{\theta=\frac{5\pi}{6}} + \left(\frac{1}{3} \cdot \frac{1}{-2} \cdot \frac{1}{\sin^2\theta}\right)\Big|_{\theta=\frac{\pi}{6}}^{\theta=\frac{5\pi}{6}}$$

$$= \boxed{0}$$

* Today: Vector fields (§ 16.1 of textbook)

Def: Let D be a region in $\mathbb{R}^2$. A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point $(x,y) \in D$ a two-dimensional vector $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$

scalar fields

Def: Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns to each point $(x,y,z) \in E$ a 3-dim vector $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

LECTURE #13

Ex 4: Describe the following vector fields by sketching these vector fields at some points and explaining in words!

(a) $F(x,y) = y\hat{\imath} - x\hat{\jmath}$

(b) $F(x,y,z) = y\hat{\imath}$

(c) $F(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

Recall the gradient vector fields

$$\boxed{\begin{array}{l} f(x,y) \rightsquigarrow \nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle \\ f(x,y,z) \rightsquigarrow \nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle \end{array}}$$

Ex 5: Find the gradient vector fields of:

(a) $f(x,y) = e^x \cdot \sin(2xy)$

(b) $f(x,y,z) = \frac{1}{2}(x^2+y^2+z^2)$

Rmk: Gradient vector fields are always perpendicular to level curves

! Hand out matching game on Vector fields

* Line Integrals (§ 16.2)

Let us be given a curve C parametrized as $\vec{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$

Then: The line integral of function $f(x,y)$ along C is defined as

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

Rmk: 1) This is independent of the parametrization of C

2) If $f(x,y) = 1$, we recover the formula for length of L.

3) If $f(x,y) \ge 0$, then $\int_C f(x,y)\,ds$ equals the area of the "fence" above C.

Likewise, if a curve C in $\mathbb{R}^3$ is given by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$, and we are given a continuous function $f(x,y,z)$, then we define

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt}$$

Rmk: You may think of both formulas as replacing ds by $\|\vec{r}\,'(t)\|dt$

LECTURE #13

Ex6: Evaluate $\int_C (5+xy^2)\,ds$, where C – the unit circle.

$C: (\cos\theta, \sin\theta),\ 0 \le \theta \le 2\pi$

So: $\int_C (5+xy^2)\,ds = \int_0^{2\pi} (5+\cos\theta\sin^2\theta)\cdot \underbrace{\sqrt{(-\sin\theta)^2+(\cos\theta)^2}}_{1}\,d\theta = 10\pi + \int_0^{2\pi} \sin^2\theta\, d(\sin\theta) = \boxed{10\pi}$

Ex7: Evaluate $\int_C 3y\,ds$, $C: \{(t^2, 2t) \mid 0 \le t \le 3\}$.

$\int_C 3y\,ds = \int_0^3 3\cdot 2t\cdot\sqrt{4t^2+4}\,dt = \int_0^3 12t\sqrt{t^2+1}\,dt \overset{u=t^2+1}{\underset{du=2tdt}{=}} \int_1^{10} 6u^{1/2}\,du = 4u^{3/2}\Big|_{u=1}^{u=10} = \boxed{4(10^{3/2}-1)}$

Ex 8: Evaluate $\int_C xe^{2yz}\,ds$, C: the segment from $(0,0,0)$ to $(1,3,2)$.

$C: (t, 3t, 2t),\ 0 \le t \le 1$

$\int_C xe^{2yz}\,ds = \int_0^1 t\cdot e^{12t^2}\cdot\sqrt{14}\,dt \overset{u=12t^2}{\underset{du=24tdt}{=}} \int_0^{12} \sqrt{14}\cdot\frac{1}{24}e^u\,du = \boxed{\frac{\sqrt{14}}{24}(e^{12}-1)}$