10/15/2019

Lecture #14

* Last time $\int_C f(x,y)\,ds$ and $\int_C f(x,y,z)\,ds$

Last time we learnt the notion of line integral of f along a curve C.

in $\mathbb{R}^2$: $\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\cdot\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ where $C: \langle x(t), y(t)\rangle$ $a\le t\le b$.

in $\mathbb{R}^3$ $\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\cdot\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$ where $C: \langle x(t), y(t), z(t)\rangle$ $a\le t\le b$.

Rmks : (1) An easy way to remember is that $ds = \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}}_{\text{integrand in the arclength formula}}\,dt$

(2) If C is a line segment from $(a,0,0)$ to $(b,0,0)$, then we recover the definite integrals from high school

* Today : Line integrals of f along C with respect to x,y,z.

in $\mathbb{R}^2$

$$\boxed{\begin{aligned}\int_C f(x,y)\,dx &= \int_a^b f(x(t),y(t))\cdot x'(t)\,dt\\ \int_C f(x,y)\,dy &= \int_a^b f(x(t),y(t))\cdot y'(t)\,dt\end{aligned}}$$

Notational: $\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$.

in $\mathbb{R}^3$:

$$\boxed{\begin{aligned}\int_C f(x,y,z)\,dx &= \int_a^b f(x(t),y(t),z(t))\cdot x'(t)\,dt\\ \int_C f(x,y,z)\,dy &= \int_a^b f(x(t),y(t),z(t))\cdot y'(t)\,dt\\ \int_C f(x,y,z)\,dz &= \int_a^b f(x(t),y(t),z(t))\cdot z'(t)\,dt\end{aligned}}$$

Notational : $\int_C P(x,y,z)\,dx + \int_C Q(x,y,z)\,dy + \int_C R(x,y,z)\,dz = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$

! To distinguish $\int_C f\,ds$, they are sometimes called line integrals wrt arc length

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Ex1: Evaluate $\int_C x\,dx + y\,dy$ along the following curves:

(a) C – line segment from $(-1,0)$ to $(1,0)$

(b) C – half of the unit circle going clockwise from $(-1,0)$ to $(1,0)$

(c) C – part of the parabola $y=x^2-1$, $-1\le x\le 1$ (going from $(-1,0)$ to $(1,0)$)

(a) $C=\{(t,0)\mid -1\le t\le 1\} \Rightarrow \int_C x\,dx+y\,dy = \int_{-1}^{1} t\,dt = \boxed{0}$

(b) $C=\{(\cos t,\sin t)\mid t$ ranges from π to $0\}$

$\Rightarrow \int_C x\,dx+y\,dy = \int_{\pi}^{0} (\cos t(-\sin t)+\sin t\cdot\cos t)\,dt = \boxed{0}$

(c) $C=\{(t,t^2-1)\mid -1\le t\le 1\} \Rightarrow \int_C x\,dx+y\,dy = \int_{-1}^{1} t\,dt + (t^2-1)\cdot 2t\,dt = \int_{-1}^{1}(2t^3-t)\,dt = \boxed{0}$

Note: We got the same answer for different 3 curves.

But this is not always the case!

Ex2: Evaluate $\int_C e^x dx + xy\,dy$ for the following curves:

(a) C – part of parabola $y=x^2$ from $(0,0)$ to $(2,4)$.

(b) C – line segment from $(0,0)$ to $(2,4)$

(c) C consists of line segment from $(0,0)$ to $(1,1)$ followed up by a line segment from $(1,1)$ to $(2,4)$.

(a) $C=\{(t,t^2)\mid 0\le t\le 2\} \Rightarrow \int_C e^x dx+xy\,dy = \int_0^2 (e^t+t\cdot t^2\cdot 2t)\,dt = \left(e^t+\frac{2}{5}t^5\right)\Big|_{t=0}^{t=2} = e^2-1+\frac{64}{5}$

(b) $C=\{(2t,4t)\mid 0\le t\le 1\} \Rightarrow \int_C e^x dx+xy\,dy = \int_0^1 (e^{2t}\cdot 2\,dt + 2t\cdot 4t\cdot 4\,dt) = \left(e^{2t}+\frac{32}{3}t^3\right)\Big|_{t=0}^{t=1}$

$= e^2-1+\frac{32}{3}$

(c) $C=C_1\sqcup C_2$, $C_1=\{(t,t)\mid 0\le t\le 1\}$, $C_2=\{(1+t,1+3t)\mid 0\le t\le 1\}$.

$\int_{C_1} e^x dx+xy\,dy = \int_0^1 (e^t+t^2)\,dt = \left(e^t+\frac{t^3}{3}\right)\Big|_{t=0}^{t=1} = e-1+\frac{1}{3}$

$\int_{C_2} e^x dx+xy\,dy = \int_0^1 e^{1+t}dt + (1+t)(1+3t)\cdot 3\,dt = \int_0^1 [e^{1+t}+3(3t^2+4t+1)]\,dt =$

$= \left(e^{1+t}+3(t^3+2t^2+t)\right)\Big|_{t=0}^{t=1} = e^2-e+3\cdot 4$

$\Rightarrow \int_C e^x dx+xy\,dy = \int_{C_1} -\text{''}- + \int_{C_2} -\text{''}- = e^2-1+\frac{1}{3}+12$

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Remark: In Ex 2 we got different integrals while varying curve.
As we will see latter on, the reason why in Ex 1 the answer was independent of the curve is b/c $\langle \underset{\text{coeff. of } dx}{\underset{\uparrow}{x}}, \underset{\text{coeff. of } dy}{\underset{\uparrow}{y}} \rangle$ is a gradient vector field of $\frac{1}{2}(x^2+y^2)$.

Remark: Note that $\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$, $\int_C f(x,y)\,dx = -\int_{-C} f(x,y)\,dx$,

$$\int_C f(x,y)\,dy = -\int_{-C} f(x,y)\,dy$$

where $-C$ denotes the same curve C but passed in the opposite direction.

Ex 3: Evaluate $\int_C xye^{yz}\,dy$, C: $x=t, y=t^2, z=t^3$, $0\le t\le 1$

▶ $$\int_0^1 t\cdot t^2\cdot e^{t^5}\cdot 2t\,dt = \int_0^1 2t^4 e^{t^5}\,dt = \frac{2}{5}e^{t^5}\Big|_{t=0}^{t=1} = \frac{2}{5}(e-1)$$ ■

* Line Integrals of Vector Fields

Def: Let $\vec{F}$ be a continuous vector field on a smooth curve C given by a vector function $\vec{r}(t)$, $a\le t\le b$.
Then the line integral of $\vec{F}$ along C is:

$$\boxed{\int_C \vec{F}\,d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt}$$

↑ also equals $\int_C \underbrace{\vec{F}(\vec{r}(t))\cdot\vec{T}}_{\mathrm{comp}_{\vec{r}\,'(t)}\vec{F}}\,ds$, $\vec{T}$ - unit tangent vector at the given point

Ex 4: Evaluate $\int_C F\,dr$, where $F(x,y,z) = x\vec{i}+y^2\vec{j}+z^3\vec{k}$, while C is given by $x=t, y=t^2, z=t^3$, $0\le t\le 1$.

▶ $$\int_C F\,dr = \int_0^1 (t\vec{i}+t^4\vec{j}+t^9\vec{k})\cdot(\vec{i}+2t\vec{j}+3t^2\vec{k})\,dt$$
$$= \int_0^1 (t+2t^5+3t^{11})\,dt = \left(\frac{t^2}{2}+\frac{t^6}{3}+\frac{t^{12}}{4}\right)\Big|_{t=0}^{t=1} = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}$$ ■

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Note that if $F = P\vec{i} + Q\vec{j} + R\vec{k}$, $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, then

$$\int_C F\,dr = \int_a^b [P(x(t),y(t),z(t))\cdot x'(t) + Q(x(t),y(t),z(t))y'(t) + R(x(t),y(t),z(t))\cdot z'(t)]\,dt$$

$$\Downarrow$$

$$\boxed{\int_C F\,dr = \int_a^b P\,dx + Q\,dy + R\,dz}$$

$\Rightarrow$ recover the same kind of line integral that we discussed in the beginning of today's class.

Ex5: Evaluate the line integral $\int_C F\,dr$ where

$F(x,y) = e^x\vec{i} - \sin(y)\vec{j}$, $C: \vec{r}(t) = t^2\vec{i} + t^3\vec{j}$, $0 \le t \le 1$

$$\left.\begin{aligned} F(x,y) &= \langle e^x, -\sin y\rangle \\ \vec{r}(t) &= \langle t^2, t^3\rangle \end{aligned}\right\} \Rightarrow \begin{aligned} F(\vec{r}(t)) &= \langle e^{t^2}, -\sin(t^3)\rangle \\ \vec{r}\,'(t) &= \langle 2t, 3t^2\rangle \end{aligned}$$

So: $\int_C F\,dr = \int_0^1 (2te^{t^2} - 3t^2\sin(t^3))\,dt = \left[e^{t^2} + \cos(t^3)\right]\Big|_{t=0}^{t=1} = e - 1 + \cos(1) - 1$

$= \boxed{e + \cos(1) - 2}$

Ex6: Evaluate $\int_C F\,dr$, where $F(x,y) = xy^2\vec{i} - x^3\vec{j}$

$C: \vec{r}(t) = t^3\vec{i} + t^4\vec{j}$, $0 \le t \le 1$.

$\int_C F\,dr = \int_0^1 \langle t^{11}, -t^9\rangle \cdot \langle 3t^2, 4t^3\rangle\,dt = \int_0^1 (3t^{13} - 4t^{12})\,dt = \left(\frac{3}{14}t^{14} - \frac{4}{13}t^{13}\right)\Big|_{t=0}^{t=1}$

$= \boxed{\frac{3}{14} - \frac{4}{13}}$

Ex7: Evaluate $\int_C F\,dr$, where $F(x,y,z) = \sin x\cdot\vec{i} + \cos y\cdot\vec{j} + xz\cdot\vec{k}$

$C: \vec{r}(t) = t^3\vec{i} - t^2\vec{j} + t\vec{k}$, $0 \le t \le 1$

$\int_C F\,dr = \int_0^1 \langle \sin(t^3), \cos(-t^2), t^4\rangle \cdot \langle 3t^2, -2t, 1\rangle\,dt$

$= \left(-\cos(t^3) + \sin(-t^2) + \frac{1}{5}t^5\right)\Big|_{t=0}^{t=1} = 1 - \cos(1) + \sin(-1) + \frac{1}{5} = \boxed{\frac{6}{5} - \cos(1) - \sin(1)}$