Lecture #16

* Last time

Last time we learned the notion of conservative vector fields and the criteria (mixed partials test) to verify whether a given vector field is conservative or not. However, if the criteria gives a positive answer, we need to find the corresponding potential by iterative integration w.r.t x, y, z.

Finally, we also learned the "FTLI", which allows to find the integrals of conservative vector fields by:

* either finding a potential
* or replacing the curve b/w the start/end-points by a simpler one and computing the line integral in a straightforward way

Ex1: Let $\vec{F} = \langle 2xy+z, x^2, x\rangle$ and C is curve given by $\vec{r}(t) = \langle \sin(t^2), \cos(t^2), \cos(t^2)\rangle$, $0 \le t \le \sqrt{\pi}$.

Evaluate $\int_C \vec{F}\, d\vec{r}$.

- In this problem, you could compute the line integral in a straightforward way (use $u=t^2$ substitution to evaluate $\int_C$) given more time.
- Let's, however, compute this integral via FTLI.

Step 1

$(2xy+z)_y = 2x = (x^2)_x$, $(2xy+z)_z = 1 = (x)_x$, $(x^2)_z = 0 = (x)_y \Rightarrow \vec{F}$ – conservative.

Step 2

Find potential to be $f(x,y,z) = x^2y + xz + C_0$ ← constant (can be ignored for FTLI!)

Step 3

Apply FTLI: $\int_C \vec{F}\, d\vec{r} = f(B) - f(A)$, where $B = \vec{r}(\sqrt{\pi}) = \langle 0,-1,-1\rangle$
$A = \vec{r}(0) = \langle 0, 1, 1\rangle$

$$\Rightarrow \int_C \vec{F}\, d\vec{r} = \boxed{0}$$

Ex 1': For $\vec{F}$ as in Ex1, compute $\int_{C'} \vec{F}\, d\vec{r}$, where C' is the circle $x^2+y^2=4$, $z=2$.

As $\vec{F}$ is conservative and C' – closed $\Rightarrow \int_{C'} \vec{F}\, d\vec{r} = 0$

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Ex 2: Let $\vec{F} = \langle 2x+3y,\ y^2+3x+e^{\sin(y)}\rangle$, C – top half of the unit circle $x^2+y^2=1$, oriented counterclockwise. Compute $\int_C \vec{F}\,d\vec{r}$.

- First, we check if $\vec{F}$ is conservative: $(2x+3y)_y = 3 = (y^2+3x+e^{\sin(y)})_x$
$$\Downarrow$$
$$\boxed{\vec{F} \text{ – conservative.}}$$

- Now, we can try to find potential f of $\vec{F}$, i.e. a function of two variables s.t. $f_x = 2x+3y$, $f_y = y^2+3x+e^{\sin(y)}$.

From the first equality we find $f(x,y) = x^2+3xy+g(y)$ ← a function of y.
But from the second equality, we get

$$y^2+3x+e^{\sin(y)} = f_y = 3x+g'(y) \Rightarrow g'(y) = y^2+\underbrace{e^{\sin(y)}}$$

↑ there is no closed formula for antiderivative of $e^{\sin(y)}$

So: Unlike all the previous examples, we know that $\vec{F}$ is conservative, but we cannot find an explicit potential.

Route #1

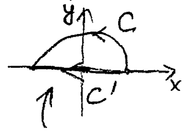
Let $g(y)$ be an antiderivative of $y^2+e^{\sin(y)}$ (it does exist for sure!).
Then $f(x,y) = x^2+3xy+g(y)$ – potential of $\vec{F}$

$$\Downarrow \text{FTLI}$$

$$\int_C \vec{F}\,d\vec{r} = f(-1,0) - f(1,0) = ((-1)^2+0+g(0)) - (1^2+0+g(0)) = \boxed{0} \quad \left(\text{so } g(0) \text{ got cancelled}\right)$$

Route #2

Replace the curve by a simpler one between the same start/end-points and compute explicitly:



$$\int_C \vec{F}\,d\vec{r} = \int_{C'} \vec{F}\,d\vec{r} = \int_1^{-1} \langle 2t, 3t+1\rangle \cdot \langle 1,0\rangle\,dt = \int_1^{-1} 2t\,dt = \boxed{0}$$

C': $(t,0)$, t goes from 1 to -1

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! Note that the FTLI guarantees that $\boxed{\int_C \vec{F}\,d\vec{r} = 0 \text{ if } \begin{array}{l} C\text{-closed path} \\ \vec{F}\text{-conservative vector field.} \end{array}}$

* Today: Green's Theorem

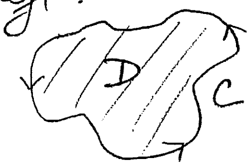
Today we shall learn another very important tool, which allows to compute $\int_C \vec{F}\,d\vec{r}$ over closed paths by reducing them to double integrals.

Theorem (Green's Theorem): Let $\vec{F} = \langle P(x,y), Q(x,y)\rangle$ be a vector field and C be a closed, positively oriented curve enclosing a region D, and assume that P, Q have continuous partials on D. Then:

$$\boxed{\int_C \vec{F}\,d\vec{r} = \iint_D (Q_x - P_y)\,dA}$$

|| C is positively oriented if walking along C in this direction, D is always on the left:

Examples:

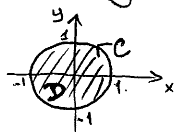


— positive orientation

← negative orientation.

! Note: If $\vec{F}$ — conservative $\Rightarrow Q_x - P_y = 0 \Rightarrow \int_C \vec{F}\,d\vec{r} = 0$ as already observed above.

Ex3: Let C be the unit circle oriented counterclockwise. Compute $\int_C (x^2+y)\,dx + (y + e^{\sin(y)} + x^2)\,dy$.



Using Green's Thm: $\int_C (x^2+y)\,dx + (y+e^{\sin(y)}+x^2)\,dy = \int_C \vec{F}\,d\vec{r} =$ ← $\vec{F} = \langle x^2+y, y+e^{\sin(y)}+x^2\rangle$

$$= \iint_{D\text{-unit disk}} (2x-1)\,dA = \int_0^{2\pi}\int_0^1 (2r\cos\theta - 1)\cdot r\,dr\,d\theta = \boxed{-\pi}$$

! Note: If C — closed path, negatively oriented, then we can reduce to the positively oriented setup above via $\int_C \vec{F}\,d\vec{r} = -\int_{-C} \vec{F}\,d\vec{r}$.

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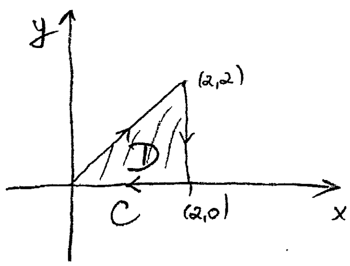
Ex 4: Let $\vec{F} = \langle \sin(x), x^2y^3 \rangle$ and C be the triangle with vertices $(0,0)$, $(2,0)$, $(2,2)$, oriented clockwise. Compute $\int_C \vec{F}\,d\vec{r}$.



▸ As this orientation is negative, applying Green's Thm we get

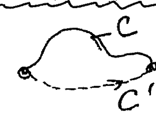
$$\int_C \vec{F}\,d\vec{r} = -\int_{-C} \vec{F}\,d\vec{r} = -\iint_D \left((x^2y^3)_x - (\sin x)_y\right) dA =$$

$$= -\iint_D 2xy^3\,dA = -\int_0^2\int_0^x 2xy^3\,dy\,dx = -\int_0^2 \left(\frac{xy^4}{2}\Big|_{y=0}^{y=x}\right) dx =$$

$$= -\int_0^2 \frac{x^5}{2}\,dx = -\frac{x^6}{12}\Big|_{x=0}^{x=2} = -\frac{64}{12} = \boxed{-\frac{16}{3}}$$

Finally, let us investigate a problem where one wants to apply the Green's Theorem, but the path C is not closed.

Hint: • Close up the path as follows:



• apply Green's Thm to evaluate $\int_{C\cup C'} \vec{F}\,d\vec{r}$

• evaluate in a straightforward way $\int_{C'} \vec{F}\,d\vec{r}$.

Ex 5: Let $\vec{F} = \langle x^3+y, y^2 \rangle$ and C be a path from $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,1)$ along straight line segments. Evaluate $\int_C \vec{F}\,d\vec{r}$.

▸ One can compute this in a straightforward way, but let us do it much faster using Green's Thm.

$(0,t)$, t goes from 1 to 0

$$\int_{C\cup C'} \vec{F}\,d\vec{r} = \iint_D [(y^2)_x - (x^3+y)_y]\,dA = -\iint_D 1\cdot dA = -\text{Area}(D) = -1.$$

$$\| $$

$$\int_C \vec{F}\,d\vec{r} + \int_{C'} \vec{F}\,d\vec{r}$$

$$\int_{C'} \vec{F}\,d\vec{r} = \int_1^0 \langle t, t^2\rangle\cdot\langle 0,1\rangle\,dt = \frac{t^3}{3}\Big|_{t=1}^{t=0} = -\frac{1}{3}$$

$$\Rightarrow \int_C \vec{F}\,d\vec{r} = \boxed{-\frac{2}{3}}$$

! It is a good exercise to compute $\int_C \vec{F}\,d\vec{r}$ in Ex5 in a straightforward way (over each line segment separately) and compare the answer to the above one.

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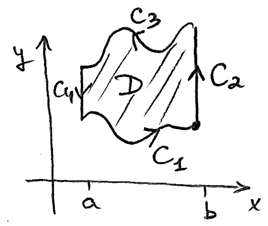
Sketch of the proof of Green's Theorem in the particular case when $Q(x,y)=0$ and $D=\{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$



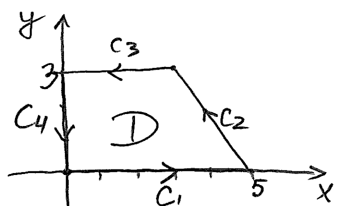
$$\left.\begin{array}{l}\int_{C_1} P(x,y)\,dx = \int_a^b P(x,g_1(x))\,dx\,, \quad \int_{C_3} P(x,y)\,dx = -\int_a^b P(x,g_2(x))\,dx \\ \int_{C_2} P(x,y)\,dx = 0, \quad \int_{C_4} P(x,y)\,dx = 0\end{array}\right\} \Rightarrow$$

$$\Rightarrow \int_{C=C_1\cup C_2\cup C_3\cup C_4} P(x,y)\,dx = \int_a^b \big(P(x,g_1(x)) - P(x,g_2(x))\big)\,dx \overset{\text{Fund. Thm. of Calculus}}{=} -\int_a^b\int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx = -\iint_D P_y\,dA$$

■

Ex 6: Evaluate $\int_C (x^3+e^{x+y}+e^{x^2})\,dx + (xy+e^{y^2}+e^{x+y})\,dy$, where

C is the curve going from $(0,0)$ to $(5,0)$ to $(3,3)$ to $(0,3)$ back to $(0,0)$ along straight line segments.



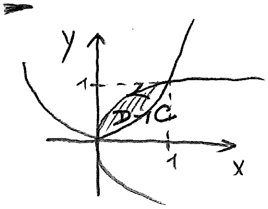
$$\int_C \overbrace{(x^3+e^{x+y}+e^{x^2})}^{P(x,y)}\,dx + \overbrace{(xy+e^{y^2}+e^{x+y})}^{Q(x,y)}\,dy \overset{\text{Green's Thm}}{=} \iint_D (y+e^{x+y}-e^{x+y})\,dA$$

$$= \iint_D y\,dA = \int_0^3\int_0^{5-\frac{2}{3}y} y\,dx\,dy = \int_0^3 y\left(5-\frac{2}{3}y\right)dy = \left(\frac{5}{2}y^2 - \frac{2}{9}y^3\right)\Big|_{y=0}^{y=3}$$

$$= \frac{5}{2}\cdot 9 - 6 = \boxed{\frac{33}{2}}$$

■

Ex 7: Let C be the boundary of the region bounded by $y=x^2$ and $x=y^2$, positively oriented. Compute $\int_C (3y+e^{2\sqrt{x}})\,dx + (x+\sin(y^3))\,dy$



$$\int_C (3y+e^{2\sqrt{x}})\,dx + (x+\sin(y^3))\,dy = \iint_D (1-3)\,dA = \iint_D -2\,dA =$$

$$= -2\int_0^1\int_{x^2}^{\sqrt{x}} 1\,dy\,dx = -2\int_0^1 (x^{1/2}-x^2)\,dx = -2\left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Big|_{x=0}^{x=1} = \boxed{-\frac{2}{3}}$$

■

! Green's Theorem may be also used other way around: reducing double integral to line integral

e.g. $\boxed{\text{Area}(D) = \int_C x\,dy = \frac{1}{2}\int_C x\,dy - y\,dx\,, \quad C = \text{boundary of } D, \text{ positively oriented}}$

Ex 8: Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The boundary C may be parametrized via $\vec{r}(t) = \langle a\cos t, b\sin t\rangle$, $0 \le t \le 2\pi$.

Thus: $\text{Area}(D) = \frac{1}{2}\int_C (x\,dy - y\,dx) = \frac{1}{2}\int_0^{2\pi} (a\cos t\cdot b\cos t - b\sin t\cdot(-a\sin t))\,dt = \frac{1}{2}\int_0^{2\pi} ab\,dt = \boxed{\pi ab}$

■