LECTURE #17

10/29/2019

* Last week (sorry for being replaced by other Lecturers)

- Explain in 10 sec why $\int_{-C} f\,ds = \int_C f\,ds$, but $\int_{-C} \vec{F}\,d\vec{r} = -\int_C \vec{F}\,d\vec{r}$
- Conservative vector fields (note the difference from the definition in the textbook as we request that components have continuous partials everywhere)
  - → Mixed partials test
  - → Algorithm to find potential function (if it exists, then it is defined up to a constant C_0)
- Fundamental Theorem of line integrals (FTLI)
  - → find the potential (can pick any of those, e.g. set $C_0=0$) and apply FTLI on the nose
  - → if the field is conservative, but no explicit formula for the potential is known, then replace the path by a simpler one between the same start/end points and evaluate it in a straightforward way. usually, a line segment

Ex 1 (see Ex 2 from Lecture #16): Let $\vec{F} = \langle 2x+3y,\ y^2+3x+e^{\sin(y)} \rangle$, C – top half of the unit circle $x^2+y^2=1$ oriented counterclockwise.

Compute $\int_C \vec{F}\,d\vec{r}$

➤ Answer: 0 See the proof in Lecture #16 Notes ∎

- Green's Theorem: $\int_{C=\partial D} \underbrace{\vec{F}}_{\langle P,Q\rangle}\,d\vec{r} = \iint_D (Q_x - P_y)\,dA$, where the boundary $C=\partial D$ of D is positively oriented
  - → If $\vec{F}$-conservative, then as we already know from FTLI $\int_{C=\partial D} \vec{F}\,d\vec{r} = 0$
  - → If C is closed path (boundary of D), but negatively oriented, then we reduce to the above via $\int_C \vec{F}\,d\vec{r} = -\int_{-C} \vec{F}\,d\vec{r}$, where $-C$ is now positively oriented.
  - → If C is not closed, then first close C by adding another curve C', so that Green's Theorem applies to $\int_{C\cup C'} \vec{F}\,d\vec{r}$, BUT then also compute $\int_{C'} \vec{F}\,d\vec{r}$ in a straightforward way (note: $\int_{C\cup C'} \vec{F}\,d\vec{r} = \int_C \vec{F}\,d\vec{r} + \int_{C'} \vec{F}\,d\vec{r}$)

Ex 2 (see Ex 5 from Lecture #16): Let $\vec{F} = \langle x^3+y, y^2 \rangle$, C be a path from $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,1)$ along straight line segments. Evaluate $\int_C \vec{F}\,d\vec{r}$.

➤ Answer: $-\frac{2}{3}$ See the proof in Lecture #16 Notes

! It is also instructive to compute this line integral in a straightforward way ∎

- Can also use Green's Theorem other way around, e.g.

$$\boxed{\text{Area}(D) = \int_C x\,dy = \frac{1}{2}\int_C x\,dy - y\,dx, \quad C=\partial D \text{ (positively oriented)}}$$

Ex 3: Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

➤ Answer: πab See the proof in Lecture #16 Notes ∎

! HAND OUT "Line Integrals in the plane" chart.

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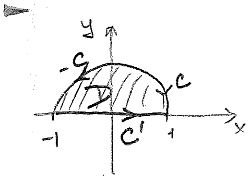
Ex 4: Let $\vec{F} = \langle y^2x + x^2,\ x^2y + x - y^{y\sin(y)} \rangle$, C - top half of $x^2+y^2=1$ oriented clockwise. Compute $\int_C \vec{F}\,d\vec{r}$.



Green's Theorem: $\int_{(-C)\cup C'} \vec{F}\,d\vec{r} = \iint_D \left[(x^2y+x-y^{y\sin(y)})_x - (y^2x+x^2)_y\right] dA = \iint_D dA = \text{Area}(D) = \frac{\pi}{2}$

C': $(t,0)$, $-1\le t\le 1 \Rightarrow \int_{C'} \vec{F}\,d\vec{r} = \int_{-1}^{1} \langle t^2, t\rangle\cdot\langle 1,0\rangle\,dt = \int_{-1}^{1} t^2\,dt = \frac{t^3}{3}\Big|_{t=-1}^{t=1} = \frac{2}{3}$

$\Rightarrow \int_{-C} \vec{F}\,d\vec{r} = \frac{\pi}{2} - \frac{2}{3} \Rightarrow \int_C \vec{F}\,d\vec{r} = -\int_{-C} \vec{F}\,d\vec{r} = \frac{2}{3} - \frac{\pi}{2}$.

- Path independence

Given a continuous vector field $\vec{F}$ with domain D, we say $\int_C \vec{F}\,d\vec{r}$ is independent of path if $\int_{C_1} \vec{F}\,d\vec{r} = \int_{C_2} \vec{F}\,d\vec{r}$ for any two paths C_1, C_2 in D with same start/end points.

It is equivalent to $\int_C \vec{F}\,d\vec{r} = 0$ for any closed path C in D ← Explain WHY!

Theorem: (a) If $\vec{F}$ is conservative, then FTLI implies that $\int_C \vec{F}\,d\vec{r}$ is indep. of path.

(b) If $\vec{F}$ is such that $\int_C \vec{F}\,d\vec{r}$ is independent of path and D is simply-connected (i.e. does not contain holes), then $\vec{F}$ is conservative.

* Today: "Curl" and "Divergence".

Def: Given a vector field $\vec{F} = \langle P, Q, R\rangle$ on $\mathbb{R}^3$, s.t. partial derivatives of P, Q, R exist, the curl of $\vec{F}$ is defined as follows:

$$\boxed{\text{curl}(\vec{F}) := \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle}$$

A simple way to remember this formula is as follows. Consider a vector differential operator

$$\boxed{\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}}$$

Compute now $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \text{curl}(\vec{F})$

So: $\boxed{\text{curl}(\vec{F}) = \vec{\nabla}\times\vec{F}}$

Def: Given a vector field $\vec{F} = \langle P, Q, R\rangle$ on $\mathbb{R}^3$ s.t. partial derivatives P_x, Q_y, R_z exist, the divergence of $\vec{F}$ is a function of 3 variables:

$$\boxed{\text{div}(\vec{F}) := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

As above: $\boxed{\text{div}(\vec{F}) = \vec{\nabla}\cdot\vec{F}}$

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Ex 5: Find the curl and divergence of $\vec{F} = xye^{z}\,\hat{\imath} + \sin(yz)\,\hat{\jmath} + xze^{y}\,\hat{k}$.

$\text{div}(\vec{F}) = \vec{\nabla}\cdot\vec{F} = ye^{z} + z\cos(yz) + xe^{y}$

$\text{curl}(\vec{F}) = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xye^{z} & \sin(yz) & xze^{y} \end{vmatrix} = (xze^{y} - y\cos(yz))\,\hat{\imath} - (ze^{y} - xye^{z})\,\hat{\jmath} - xe^{z}\,\hat{k}$

> **Theorem**: (1) $\text{curl}(\nabla f) = 0$ for any $f(x,y,z)$ with continuous second order partial derivatives
>
> (2) If $\vec{F}$ is a vector field defined on all $\mathbb{R}^3$, whose components have continuous partial derivatives and $\text{curl}(\vec{F}) = \vec{0}$, then $\vec{F}$ is conservative ($\vec{F} = \nabla f$ for some f)

! This follows immediately from Lecture #15 by noticing that $\text{curl}(\vec{F}) = \vec{0}$ if and only if $\vec{F}$ satisfies mixed partials test.

Ex 6: Determine if the following vector field is conservative or not. If yes – find potential

(a) $\vec{F} = \langle e^{y}\cos x,\ xye^{z},\ z\sin y \rangle$

(b) $\vec{F} = \langle 1 + e^{y} + ze^{x},\ xe^{y},\ e^{x} \rangle$

(a) The coefficient of $\hat{\imath}$ in $\text{curl}(\vec{F}) = \vec{\nabla}\times\vec{F}$ is $z\cos y - xye^{z} \neq 0 \Rightarrow \vec{F}$ – not conservative.

(b) $\text{curl}(\vec{F}) = 0 \Rightarrow$ conservative. Easy to determine potential:

$$f(x,y,z) = ze^{x} + xe^{y} + x + C_0 \leftarrow \text{constant}$$

Ex 7: Compute divergence of $\text{curl}(\vec{F})$ from Ex 5.

$\text{div}(\text{curl}(\vec{F})) = ze^{y} - ze^{y} + xe^{z} - xe^{z} = 0$

This answer is not accidental:

> **Theorem**: If $\vec{F} = \langle P, Q, R \rangle$ is a vector field on $\mathbb{R}^3$ and P, Q, R have continuous second order partial derivatives, then $\boxed{\text{div}(\text{curl}(\vec{F})) = 0}$

$$\text{div}(\text{curl}(\vec{F})) = \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} = 0$$

! The opposite is also true, i.e. if $\text{div}(\vec{G}) = 0$, then $\vec{G} = \text{curl}(\vec{F})$ for some $\vec{F}$

Ex 8: Determine whether or not a vector field $\vec{G}$ is a curl of some $\vec{F}$.

(a) $\vec{G} = \langle x\sin z,\ y^2 + xz,\ x + \cos z \rangle$

(b) $\vec{G} = \langle -y, z, y \rangle$

(a) $\text{div}\,\vec{G} = \sin z + 2y - \sin z = 2y \neq 0 \Rightarrow \vec{G}$ is not a curl.

(b) $\text{div}\,\vec{G} = 0 + 0 + 0 = 0 \Rightarrow \vec{G}$ is a curl