LECTURE #18

* Organizational: PLEASE fill out the midterm surveys by Friday night.

* Last time: Curl and Divergence ← we shall need them next week and later on.

To evoke the definition, let us do a simple exercise.

Ex 1: Prove $\text{div}(f\cdot\vec{F}) = f\cdot \text{div}(\vec{F}) + \vec{F}\circ\nabla f$

$\vec{F} = \langle P, Q, R\rangle \Rightarrow f\vec{F} = \langle fP, fQ, fR\rangle$

So: $\text{div}(f\vec{F}) = \frac{\partial(fP)}{\partial x} + \frac{\partial(fQ)}{\partial y} + \frac{\partial(fR)}{\partial z} = (f_x\cdot P + f\cdot P_x) + (f_y\cdot Q + f\cdot Q_y) + (f_z\cdot R + f\cdot R_z)$

$= f\cdot(P_x + Q_y + R_z) + (Pf_x + Qf_y + Rf_z) = f\cdot \text{div}\,\vec{F} + \vec{F}\circ\nabla f$ ∎

*Today: Parametric surfaces (§16.6 of the textbook)

Similar to curves, we can describe a surface by a vector function $\vec{r}(u,v)$:

$$\boxed{\vec{r}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{k}}$$ ← vector-valued function defined on $D \subseteq \mathbb{R}^2_{u,v}$

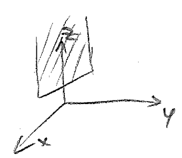
Def: The set of all points $(x,y,z)\in\mathbb{R}^3$ such that $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ for some $(u,v)\in D$ is called a parametric surface.

Ex 2: Identify and sketch the following surfaces with vector equation:

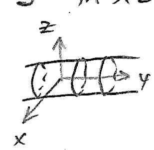
(a) $\vec{r}(u,v) = (1-2u)\,\hat{\imath} + 2u\,\hat{\jmath} + e^{v^2}\,\hat{k}$, $D = \{(u,v) \mid u\in\mathbb{R},\ v\geq 0\}$

(b) $\vec{r}(u,v) = 5\sin(u)\cdot\hat{\imath} - 10v\cdot\hat{\jmath} + 5\cos(u)\cdot\hat{k}$

(a) part of the plane $x+y=1$ with $z\geq 1$



(b) cylinder whose base is a circle of radius 5 in xz-plane centered at the origin and with axis being the y-axis



∎

Def: If we keep u constant by putting $u = u_0$, then $\vec{r}(u_0, v)$ defines a curve C_1 on surface S. Similar, if we keep v constant by putting $v = v_0$, then $\vec{r}(u, v_0)$ defines a curve C_2 on S. These are the grid curves.

Ex 3: Describe grid curves for two surfaces from Ex 2.

(a) Get "vertical half-lines" and "horizontal lines" in our half-plane.

(b) Get straight lines // y-axis and circles parallel to xz-plane. ∎

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Ex 4: Find a parametric representation of the plane passing through the point $P_0(x_0, y_0, z_0)$ and containing two non-parallel vectors $\vec{a}=\langle a_1, a_2, a_3\rangle$, $\vec{b}=\langle b_1, b_2, b_3\rangle$

$$\begin{cases} x = x_0 + a_1\cdot u + b_1\cdot v \\ y = y_0 + a_2\cdot u + b_2\cdot v \\ z = z_0 + a_3\cdot u + b_3\cdot v \end{cases}$$ ■

Ex 5: Find a parametric representation of the cylinder $x^2+y^2=9$, $-2\le z\le 5$.

$$\begin{cases} x = 3\cos u \\ y = 3\sin u \\ z = v \end{cases} \quad \text{with} \quad \begin{matrix} 0\le u\le 2\pi \\ -2\le v\le 5 \end{matrix}$$ ■

Ex 6: Find a parametric representation of the graph of a function $f(x,y)$ on $\mathbb{R}^2$.

$$\begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases}$$ ■

Q: How to parametrize the surface given by $x = f(y,z)$?

Ex 7: Find a parametric representation of the top half (i.e. $z\ge 0$) of the cone $z^2=16(x^2+y^2)$.

One way: $\left.\begin{matrix} z^2=16(x^2+y^2) \\ z\ge 0 \end{matrix}\right\} \Rightarrow z = 4\sqrt{x^2+y^2} \xRightarrow{\text{Ex 6}} \begin{cases} x=u \\ y=v \\ z=4\sqrt{u^2+v^2} \end{cases}, \quad (u,v)\in\mathbb{R}^2$.

Alternative way: $x = r\cos\theta$, $y = r\sin\theta$, $z = 4r$, $(r\ge 0,\ 0\le\theta\le 2\pi)$ ■

Ex 8: Find a parametric representation of the elliptic paraboloid $y=2-x^2-2z^2$.

$$\begin{cases} x = u \\ y = 2-u^2-2v^2 \\ z = v \end{cases}, \quad (u,v)\in\mathbb{R}^2$$ ■

Ex 9 (Surface of revolution): Let S be obtained by rotating the curve $y=f(x)$, $a\le x\le b$ (assume $f(x)\ge 0$) around x-axis. Find a parametric equation of S.

Taking a point $(x_0, y_0=f(x_0))$ in the xy-plane and rotating it by angle θ around x-axis, we clearly get a point $(x_0, f(x_0)\cdot\cos\theta, f(x_0)\cdot\sin\theta)$.

So: S may be parametrized via

$$\begin{cases} x = u \\ y = f(u)\cdot\cos(v) \\ z = f(u)\cdot\sin(v) \end{cases}, \quad \begin{matrix} a\le u\le b \\ 0\le v\le 2\pi \end{matrix}$$ ■

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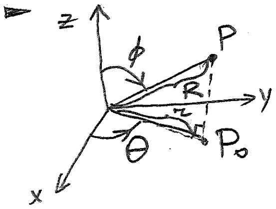
Ex 10: Find a parametric equation of the sphere $x^2+y^2+z^2=R^2$.



$z = R\cos\phi$

$x = r\cos\theta,\ y = r\sin\theta$ and $r = R\sin\phi$

So: $\begin{cases} x = R\sin\phi\cos\theta \\ y = R\sin\phi\sin\theta \\ z = R\cos\phi \end{cases}$ with $\begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{array}$

note: π not 2π!

Ex 11: Describe grid curves for the surfaces from Ex 4-10

! Hand out the matching game (quite hard!).

* Today: Tangent planes to parametric surfaces.

Want: Find the tangent plane to the parametric surface S traced out by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ at the point $P_0 = \vec{r}(u_0,v_0)$.

As we already know, the tangent plane contains tangent vectors at P_0 to various curves on S passing through P_0. In particular, we have tangent vectors to the grid curves C_1 and C_2 passing through P_0.

Grid Curve C_1: traced out by $\vec{r}(u_0,v) \Rightarrow$ tangent vector is $\frac{\partial}{\partial v}\vec{r}(u_0,v)\big|_{v=v_0}$, which equals
$$\boxed{\vec{r}_v = \left\langle \frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0)\right\rangle}$$

Grid Curve C_2: traced out by $\vec{r}(u,v_0) \Rightarrow$ tangent vector is $\frac{\partial}{\partial u}\vec{r}(u,v_0)\big|_{u=u_0}$ which equals
$$\boxed{\vec{r}_u = \left\langle \frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0)\right\rangle}$$

Def: Surface S is smooth if $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ for any P_0 on S

UPSHOT: The tangent plane to S at P_0 is uniquely determine by:
1) it contains P_0!
2) $\vec{n} := \vec{r}_u \times \vec{r}_v$ is a normal vector

Ex 12: (a) Describe the surface S, parametrized $\vec{r}(u,v) = u^2\hat{i} + 2u\sin(v)\hat{j} + u\cos(v)\hat{k}$
(b) Find the tangent plane to S at the point $P_0 = \vec{r}(1,0)$.

(a) Note: $(2u\sin v)^2 + 4\cdot(u\cos v)^2 = 4u^2 \Rightarrow$ any point $(x,y,z)\in S$ satisfies $y^2+4z^2=4x$, the opposite is also clear. So: S is an elliptic paraboloid.

(b) $\left.\begin{array}{l} \vec{r}_u = \langle 2u, 2\sin v, \cos v\rangle \Rightarrow \vec{r}_u(1,0) = \langle 2,0,1\rangle \\ \vec{r}_v = \langle 0, 2u\cos v, -u\sin v\rangle \Rightarrow \vec{r}_v(1,0) = \langle 0,2,0\rangle \end{array}\right\} \Rightarrow \vec{n} = \vec{r}_u\times\vec{r}_v = \langle -2,0,4\rangle \Bigg\} \Rightarrow$

$P_0 = \vec{r}(1,0) = (1,0,1)$

$\Rightarrow$ Tangent Plane: $-2(x-1)+4(z-1)=0$.