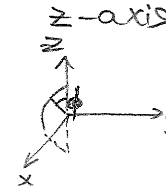
LECTURE #19 11/05/2019

* Last time : Parametric surface

- First, let us deduce the parametric equation of the sphere S $x^2+y^2+z^2=R^2$ in a different way, viewing S as a surface of revolution:

  S is obtained by revolving half-circle C : $\begin{cases} x^2+z^2=R^2 \\ x\geq 0\end{cases}$ in xz-plane around z-axis



  In xz-plane, C is parametrized $z=R\cos\phi$, $x=R\sin\phi$

  Revolving such a point by θ counterclockwise around z-axis produces a point:

$$\boxed{x=R\sin\phi\cos\theta,\quad y=R\sin\phi\sin\theta,\quad z=R\cos\phi \quad \text{with } 0\leq\phi\leq\pi,\ 0\leq\theta\leq 2\pi}$$

- For a parametric surface S: $\vec{r}(u,v)$, we concluded last time that the tangent plane to S at the point P_0 corresponding to $\vec{r}(u_0,v_0)$ is the plane containing P_0 and with the normal vector being $\vec{r}_u\times\vec{r}_v=:\vec{n}$.

Ex1: Do Ex 12 from Lecture #18.

- Ex2: Determine grid curves on the sphere $x^2+y^2+z^2=R^2$ parametrized as above.

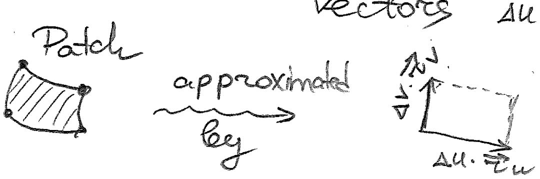
* Today: Surface Area

Def: If a smooth parametric surface is given by $\vec{r}(u,v)$, $(u,v)\in D$, and S is covered just once as (u,v) ranges through D, then the surface area of S is

$$\boxed{A(S):=\iint_D \|\vec{r}_u\times\vec{r}_v\|\,dA}$$

Rmk1: Let us explain where this formula is coming from.

Looking at the grid curves on S, we see that S is divided into small "patches". Every such patch can be approximated by a parallelogram with sides determined by vectors $\Delta u\cdot\vec{r}_u$ and $\Delta v\cdot\vec{r}_v$



But the area of such parallelogram is the magnitude of cross product

$$\|(\Delta u\cdot\vec{r}_u)\times(\Delta v\cdot\vec{r}_v)\|=\Delta u\cdot\Delta v\cdot\|\vec{r}_u\times\vec{r}_v\|$$

Rmk 2: Compare to the arc-length formula $L(C)=\int_a^b\|\vec{r}\,'(t)\|dt$.

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* Today: Surface integrals of functions.

Def: If a smooth parametric surface S is given by $\vec{r}(u,v)$, $(u,v)\in D$, and S is covered just once as (u,v) ranges through D, the surface integral of $f(x,y,z)$ over the surface S is

$$\boxed{\iint_S f(x,y,z)\,dS := \iint_D f(\vec{r}(u,v))\cdot\|\vec{r}_u\times\vec{r}_v\|\,dA}$$

Rmk 3: The motivation for this formula is completely analogous to the one from Rmk 1. Moreover, for $f(x,y,z)=1$, we just recover $A(S)$!

Rmk 4: One way to view the surface integral is as follows. Consider the sheet of aluminium foil that has shape of a surface S and the density at point (x,y,z) is $\rho(x,y,z)$. Then:

1) the total mass of the sheet is $m=\iint_S \rho(x,y,z)\,dS$

2) the center of mass is $\left(\frac{1}{m}\iint_S x\cdot\rho(x,y,z)\,dS,\ \frac{1}{m}\iint_S y\cdot\rho(x,y,z)\,dS,\ \frac{1}{m}\iint_S z\rho(x,y,z)\,dS\right)$

Ex 3: (a) Find the surface area of a sphere $S: x^2+y^2+z^2=R^2$

(b) Find the surface integral $\iint_S x^2\,dS$, where S is as in (a).

► Parametrize S as in the beginning of the class:

$\vec{r}(\phi,\theta)=\langle R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi\rangle$

Compute $\vec{r}_\phi$ and $\vec{r}_\theta$:

$\vec{r}_\phi=\langle R\cos\phi\cos\theta,\ R\cos\phi\sin\theta,\ -R\sin\phi\rangle$

$\vec{r}_\theta=\langle -R\sin\phi\sin\theta,\ R\sin\phi\cos\theta,\ 0\rangle$

Compute $\boxed{\vec{r}_\phi\times\vec{r}_\theta = R^2\sin^2\phi\cos\theta\cdot\hat{\imath}+R^2\sin^2\phi\sin\theta\cdot\hat{\jmath}+R^2\sin\phi\cos\phi\cdot\hat{k}}$

Finally, evaluate the magnitude $\|\vec{r}_\phi\times\vec{r}_\theta\|$:

$\|\vec{r}_\phi\times\vec{r}_\theta\|=\sqrt{R^4\sin^4\phi\cos^2\theta+R^4\sin^4\phi\sin^2\theta+R^4\sin^2\phi\cos^2\phi}=\sqrt{R^4\sin^4\phi+R^4\sin^2\phi\cos^2\phi}$

$=\sqrt{R^4\sin^2\phi}=R^2|\sin\phi| \overset{0\le\phi\le\pi}{=} \boxed{R^2\sin\phi}$

(a) $A(S)=\iint_D R^2\sin\phi\,dA_{\phi,\theta}=\int_0^\pi\int_0^{2\pi}R^2\sin\phi\,d\theta\,d\phi=\int_0^\pi 2\pi R^2\sin\phi\,d\phi=\boxed{4\pi R^2}$

(b) $\iint_S x^2\,dS=\iint_D R^4\sin^3\phi\cos^2\theta\,dA_{\phi,\theta}=R^4\cdot\int_0^\pi\sin^3\phi\,d\phi\cdot\int_0^{2\pi}\cos^2\theta\,d\theta$

But: $\int_0^{2\pi}\cos^2\theta\,d\theta=\int_0^{2\pi}\frac{1+\cos(2\theta)}{2}\,d\theta=\pi$

$\int_0^\pi\sin^3\phi\,d\phi=\int_0^\pi\sin^2\phi\,d(-\cos\phi)=\int_0^\pi(1-\cos^2\phi)\,d(-\cos\phi)\overset{u=-\cos\phi}{=}\int_{-1}^{1}(1-u^2)\,du=\left(u-\frac{u^3}{3}\right)\Big|_{u=-1}^{u=1}=\frac{4}{3}$

So: $\boxed{\iint_S x^2\,dS=\frac{4}{3}\pi R^4}$

LECTURE #19

Special case: graphs of functions

A quite frequent example of a surface is a graph of a function in 2 variables. For example, if S is a graph of $f(x,y)$, $(x,y)\in D$, then a natural parametrization of S is:

$$\boxed{\vec{r}(u,v) = u\cdot\hat{i} + v\cdot\hat{j} + f(u,v)\cdot\hat{k},\quad (u,v)\in D}$$

Then: $\left.\begin{array}{l}\vec{r}_u = \langle 1, 0, f_u\rangle \\ \vec{r}_v = \langle 0, 1, f_v\rangle\end{array}\right\} \Rightarrow \boxed{\vec{r}_u\times\vec{r}_v = \langle -f_u, -f_v, 1\rangle} \Rightarrow \boxed{\|\vec{r}_u\times\vec{r}_v\| = \sqrt{1+f_u^2+f_v^2}}$

Therefore:

$$\boxed{\begin{array}{l} A(S) = \iint\limits_D \sqrt{1+(f_u)^2+(f_v)^2}\, dA_{u,v} \\ \iint\limits_S g(x,y,z)\,dS = \iint\limits_D g(u,v,f(u,v))\cdot\sqrt{1+(f_u)^2+(f_v)^2}\,dA \end{array}}$$

Rmk 5: Similar formulas apply when S is a graph of f-n in x,z or y,z.

Ex 4: Find the area of the part S of paraboloid $y=x^2+z^2$ that lies within the cylinder $x^2+z^2=16$.

Parametrize S via $\vec{r}(u,v) = \langle u, u^2+v^2, v\rangle$ with domain $D=\{(u,v)\,|\,u^2+v^2\le 16\}$.

← graph of $f(x,z)=x^2+z^2$

So: $A(S) = \iint\limits_D \sqrt{1+(2u)^2+(2v)^2}\,dA = \iint\limits_{D:\,u^2+v^2\le 16} \sqrt{1+4(u^2+v^2)}\,dA \xrightarrow[\text{coordinates}]{\text{Polar}}$

$= \int_0^{2\pi}\int_0^4 \sqrt{1+4r^2}\; r\,dr\,d\theta \xrightarrow[du=8r\,dr]{u=1+4r^2} 2\pi\cdot\int_1^{65} u^{1/2}\cdot\frac{du}{8} = 2\pi\cdot\frac{1}{8}\cdot\frac{2}{3}u^{3/2}\Big|_{u=1}^{u=65} =$

$= \boxed{\frac{\pi}{6}\left(65^{3/2}-1\right)}$

Rmk 6: If we use polar coordinates in the parametrization of S in the very beginning, then we do not need this extra factor r!

Ex 5: Recalculate $A(S)$ from Ex 4 using another (polar) parametrization: $\vec{r}(r,\theta) = \langle r\cos\theta, r^2, r\sin\theta\rangle$, $D=\{(r,\theta)\,|\,0\le r\le 4,\ 0\le\theta\le 2\pi\}$.

$\left.\begin{array}{l}\vec{r}_r = \langle\cos\theta, 2r, \sin\theta\rangle \\ \vec{r}_\theta = \langle -r\sin\theta, 0, r\cos\theta\rangle\end{array}\right\} \Rightarrow \vec{r}_r\times\vec{r}_\theta = \langle 2r^2\cos\theta, -r, 2r^2\sin\theta\rangle$

$\Downarrow$

$\|\vec{r}_r\times\vec{r}_\theta\| = \sqrt{4r^4\cos^2\theta + r^2 + 4r^4\sin^2\theta} = r\cdot\sqrt{1+4r^2}$

So: $A(S) = \iint\limits_{\substack{0\le r\le 4\\ 0\le\theta\le 2\pi}} r\sqrt{1+4r^2}\,dA_{r,\theta}$ ← it is exactly the same integral as above, hence, it equals $\frac{\pi}{6}(65^{3/2}-1)$.

However: the factor r in front of $\sqrt{1+4r^2}$ is already there! ∎