* Last time: Surface Area + Surface integrals of functions

Ex 0: Do Ex 4&5 from Lecture #19.

* Today: Flux (aka surface integrals of vector fields)

Idea: While $\iint_S f(x,y,z)\,dS$ is reminiscent of $\int_C f(x,y,z)\,ds$, the flux is gonna be reminiscent of $\int_C \vec{F}\,d\vec{r}$ in some sense.

However, even to give the definition, we need to restrict our attention only to a certain class of surfaces, called oriented. Note that there are 2 unit normal vectors to S at any point P on S.

Def: If it is possible to choose a unit normal vector $\vec{n}$ at every point $(x,y,z)\in S$ so that $\vec{n}$ varies continuously over S, then S is called an oriented surface and the given choice of $\vec{n}$ provides S with an orientation.

Rmk: Not every surface is oriented, but every oriented surface has exactly 2 orientations.

Def: If $\vec{F}$ is a continuous vector field defined on an oriented surface S with an orientation defined by the unit vector $\vec{n}$ (at each point), then the flux of $\vec{F}$ across S (aka surface integral of $\vec{F}$ across S) is

$$\boxed{\iint_S \vec{F}\,d\vec{S} := \iint_S \vec{F}\cdot\vec{n}\,dS}$$

dot-product, hence, a function on S, which we know how to integrate from last lecture.

Key Step: Find $\vec{n}$!

There are two basic examples we will treat:

Example 1: S is a graph of $f(x,y)$ with $(x,y)\in D$

Parametrize S via $\vec{r}(u,v) = \langle u, v, f(u,v)\rangle$, $(u,v)\in D$

Last time: $\vec{r}_u \times \vec{r}_v = \langle -f_u, -f_v, 1\rangle$ - normal (but not unit!) vector

Hence unit normal vectors are $\pm \dfrac{-f_u\cdot\vec{i} - f_v\cdot\vec{j} + \vec{k}}{\sqrt{1+f_u^2+f_v^2}}$

Terminology: "Upward orientation" refers to $\boxed{\vec{n} = \dfrac{\langle -f_u, -f_v, 1\rangle}{\sqrt{1+f_u^2+f_v^2}}}$

Example 2: Smooth parametric surface given by $\vec{r}(u,v)$, $(u,v)\in D$

Then unit normal vectors are $\boxed{\pm\dfrac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|}}$

Def: For a closed surface (i.e. boundary of a solid E), the positive orientation is the one where the normal vector points outside of E, while the inward-pointing normals give the negative orientation.

Back to Example 2: $\vec{n} = \pm \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$, so that

$$\iint_S \vec{F} d\vec{S} = \iint_S \vec{F} \cdot \frac{\pm \vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} dS = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{(\pm \vec{r}_u \times \vec{r}_v)}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}} \cdot \cancel{\|\vec{r}_u \times \vec{r}_v\|} dA$$

So: $\boxed{\iint_S \vec{F} d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\pm \vec{r}_u \times \vec{r}_v) dA}$ $\leftarrow$ i.e. denominator $\|\vec{r}_u \times \vec{r}_v\|$ disappeared!

Ex1: Find the flux of $\vec{F}(x,y,z) = \langle 3z, 3y, 3x \rangle$ across the sphere $S: x^2+y^2+z^2=R^2$ with the positive orientation.

$\vec{r}(\phi,\theta) = \langle R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi \rangle$

Last time: $\boxed{\vec{r}_\phi \times \vec{r}_\theta = \langle R^2\sin^2\phi\cos\theta, R^2\sin^2\phi\sin\theta, R^2\sin\phi\cos\phi \rangle}$

But: $\boxed{\vec{F}(\vec{r}(\phi,\theta)) = \langle 3R\cos\phi, 3R\sin\phi\sin\theta, 3R\sin\phi\cos\theta \rangle}$ $\Rightarrow$

$$\Rightarrow \boxed{\begin{aligned}\vec{F}(\vec{r}(\phi,\theta)) \cdot (\vec{r}_\phi \times \vec{r}_\theta) &= 3R^3(\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta + \sin^2\phi\cos\phi\cos\theta) \\ &= 3R^3(2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\end{aligned}}$$

Also we need to decide whether $\vec{r}_\phi \times \vec{r}_\theta$ points inwards or outside the ball bounded by S. It suffices to check this at any point. E.g. for $\phi = \frac{\pi}{2} = \theta$, $\vec{r}_\phi \times \vec{r}_\theta = \langle 0, R^2, 0 \rangle$ – pointing outside (explain this!)

So: $\displaystyle\iint_S \vec{F} d\vec{S} = \int_0^\pi \int_0^{2\pi} 3R^3(2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta) d\theta d\phi$

But: $\displaystyle\int_0^{2\pi} \cos\theta d\theta = 0, \quad \int_0^{2\pi} \sin^2\theta d\theta = \int_0^{2\pi} \frac{1-\cos(2\theta)}{2} d\theta = \pi$ $\Rightarrow$

$$\Rightarrow \iint_S \vec{F} d\vec{S} = 3\pi R^3 \underbrace{\int_0^\pi \sin^3\phi d\phi}_{=4/3 \text{ by last time}} = \boxed{4\pi R^3}$$

! Each time you need to determine whether $\dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ or $\dfrac{-\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ determine the orientation.

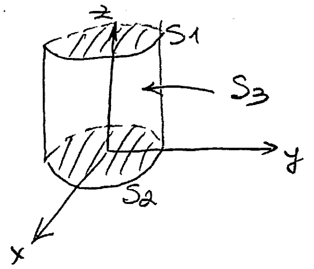
Thus: the strategy to compute flux $\iint_S \vec{F} d\vec{S}$ is:

1) Split S into several parts, parametrize each of those
2) Compute $\vec{r}_u \times \vec{r}_v$ and decide if you take it with + or − sign.
3) Evaluate the dot-product $\vec{F}(\vec{r}(u,v)) \cdot (\pm \vec{r}_u \times \vec{r}_v)$
4) Compute the double integral!

Rmk: In some simple cases one may simplify this strategy by explicitly finding $\vec{n}$.

Lecture #20

Ex2: Find the flux of the vector field $\vec{F}(x,y,z) = \langle x, y, z \rangle$ over a cylinder given by $x^2+y^2=9$, $0 \le z \le 5$ together with its top and bottom. The orientation is chosen to be positive.



We split the entire surface S into 3 parts:

S_1 – the disk $x^2+y^2 \le 9$, $z=5$ on top

S_2 – the disk $x^2+y^2 \le 9$, $z=0$ on the bottom

S_3 – side part given by $x^2+y^2=9$, $0 \le z \le 5$.

Clearly: $\iint_S \vec{F}\, dS = \iint_{S_1} \vec{F}\, dS + \iint_{S_2} \vec{F}\, dS + \iint_{S_3} \vec{F}\, dS$ and we need to compute each of these 3 fluxes.

Flux across S_1

1st way: Parametrize S_1 via $\vec{r}(u,v) = \langle u\cos v, u\sin v, 5 \rangle$ $\quad 0 \le u \le 3$, $0 \le v \le 2\pi$

Then $\left.\begin{aligned} \vec{r}_u &= \langle \cos v, \sin v, 0 \rangle \\ \vec{r}_v &= \langle -u\sin v, u\cos v, 0 \rangle \end{aligned}\right\} \Rightarrow \vec{r}_u \times \vec{r}_v = \vec{i}\cdot 0 - \vec{j}\cdot 0 + \vec{k}\cdot(u\cos^2 v + u\sin^2 v)$

$= u\cdot\vec{k}$ and from picture it's clear $u\vec{k}$ looks outside!

Hence: $\iint_{S_1} \vec{F}\, dS = \int_0^{2\pi}\int_0^3 \langle u\cos v, u\sin v, 5 \rangle \cdot \langle 0, 0, u \rangle\, du\, dv = \int_0^{2\pi}\int_0^3 5u\, du\, dv = \boxed{45\pi}$

2nd way: Let us provide an alternative way to compute $\iint_{S_1} \vec{F}\, dS$.

Just from the picture it is clear that $\vec{n} = \vec{k}$ on $S_1 \Rightarrow \vec{F}\cdot\vec{n} = 5$ on S_1

$\Rightarrow \iint_{S_1} \vec{F}\, dS = \iint_{S_1} 5\, dS = 5 \cdot \underbrace{A(S_1)}_{\text{Surface Area of } S_1 \text{ = disk of radius 3}} = 5\cdot\pi\cdot 3^2 = 45\pi$ ← Got the same answer.

Flux across S_2

1st way: Parametrize S_1 via $\vec{r}(u,v) = \langle u\cos v, u\sin v, 0 \rangle$ $\quad 0 \le u \le 3$, $0 \le v \le 2\pi$

As above, we get $\vec{r}_u \times \vec{r}_v = u\cdot\vec{k}$, but looking at the picture we actually see that $u\vec{k}$ points inwards the solid $\Rightarrow$ need to take $-u\cdot\vec{k}$.

$\Rightarrow \iint_{S_2} \vec{F}\, dS = \int_0^{2\pi}\int_0^3 \underbrace{\langle u\cos v, u\sin v, 0 \rangle \cdot \langle 0, 0, -u \rangle}_{0}\, du\, dv = \boxed{0}$

2nd Way: Looking at picture $\vec{n} = -\vec{k}$ on $S_2 \Rightarrow \vec{F}\cdot\vec{n} = 0$ on $S_2 \Rightarrow \iint_{S_2} \vec{F}\, dS = 0$

Lecture #20

Flux across S_3

1st way Parametrize S_3 via $\vec{r}(u,v) = \langle 3\cos v, 3\sin v, u\rangle$ $\quad 0 \le u \le 5$, $0 \le v \le 2\pi$.

$$\left.\begin{array}{l}\vec{r}_u = \langle 0,0,1\rangle \\ \vec{r}_v = \langle -3\sin v, 3\cos v, 0\rangle\end{array}\right\} \Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -3\sin v & 3\cos v & 0 \end{vmatrix} = -3\cos v \cdot \vec{i} - 3\sin v \cdot \vec{j} = \langle -3\cos v, -3\sin v, 0\rangle$$

Now we have to decide whether we pick $\vec{r}_u \times \vec{r}_v$ or $-\vec{r}_u \times \vec{r}_v$. We need a vector which points outwards. It suffices to check at any point on S_3. For example when $u=0, v=0$, we get $\vec{r}_u \times \vec{r}_v = \langle -3,0,0\rangle$, while looking back at the picture we see that this points inwards

So: We need to take $-\vec{r}_u \times \vec{r}_v = \langle 3\cos v, 3\sin v, 0\rangle$

Thus: $$\iint_{S_3} \vec{F}\,d\vec{S} = \int_0^{2\pi}\int_0^5 \underbrace{\langle 3\cos v, 3\sin v, u\rangle \cdot \langle 3\cos v, 3\sin v, 0\rangle}_{9}\,du\,dv = \boxed{90\pi}$$

2nd Way Looking at the picture, it is clear that $\vec{n}$ is always parallel to xy-plane and is explicitly given by $\vec{n} = \langle \frac{x}{3}, \frac{y}{3}, 0\rangle$ at the point $(x,y,z) \Rightarrow \vec{F}\cdot\vec{n} = \frac{x^2+y^2}{3} = 3$ on S_3

Hence: $$\iint_{S_3} \vec{F}\,d\vec{S} = \iint_{S_3} 3\,dS = 3\cdot A(S_3) = 3\cdot 5\cdot(2\pi\cdot 3) = 90\pi$$

Summarizing all the above, we see that

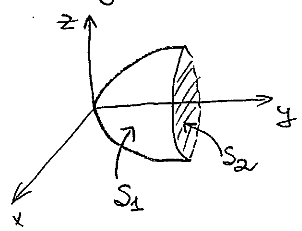
$$\iint_S \vec{F}\,d\vec{S} = 45\pi + 0 + 90\pi = \boxed{135\pi}$$

$\blacksquare$

Remark: We on purpose illustrated two approaches:
- 1st Way is the most canonical
- 2nd Way is sometimes easier (as we saw).

Lecture #20

Ex 3: Let $\vec{F}(x,y,z) = \langle 0, y, -z\rangle$. Find the flux of $\vec{F}$ across the positively oriented S, which consists of the paraboloid $y = x^2+z^2$, $0 \le y \le 1$ and the disk $x^2+z^2 \le 1$, $y=1$.



This surface S consists of two parts: S_1 and S_2

- S_1 - part of the paraboloid $y = x^2+z^2$, $0 \le y \le 1$
- S_2 - disk $x^2+z^2 \le 1$, $y = 1$.

So: $\iint_S \vec{F}\,d\vec{S} = \iint_{S_1} \vec{F}\,d\vec{S} + \iint_{S_2} \vec{F}\,d\vec{S}$

Flux across S_2

Parametrize S_2 via $\vec{r}(u,v) = \langle u\cos v, 1, u\sin v\rangle$, $\begin{matrix} 0 \le u \le 1 \\ 0 \le v \le 2\pi \end{matrix}$

$$\left.\begin{matrix} \vec{r}_u = \langle \cos v, 0, \sin v\rangle \\ \vec{r}_v = \langle -u\sin v, 0, u\cos v\rangle \end{matrix}\right\} \Rightarrow \vec{r}_u \times \vec{r}_v = -u\cdot\vec{j}$$

But looking at the picture, it is clear that to get a vector pointing outwards, we need to take $-\vec{r}_u \times \vec{r}_v = u\cdot\vec{j}$.

Hence: $\iint_{S_2} \vec{F}\,d\vec{S} = \int_0^{2\pi}\int_0^1 \langle 0, 1, -u\sin v\rangle \cdot \langle 0, u, 0\rangle\,du\,dv = \boxed{\pi}$

Note: We could as in Ex2 immediately notice that $\vec{n} = \vec{j} \Rightarrow \vec{F}\cdot\vec{n} = 1$ on S_2

$$\Rightarrow \iint_{S_2} \vec{F}\,d\vec{S} = A(S_2) = \pi.$$

Flux across S_1

Parametrize S_1 via $\vec{r}(u,v) = \langle u, u^2+v^2, v\rangle$, (u,v) is subject to $u^2+v^2 \le 1$.

$$\left.\begin{matrix} \vec{r}_u = \langle 1, 2u, 0\rangle \\ \vec{r}_v = \langle 0, 2v, 1\rangle \end{matrix}\right\} \Rightarrow \vec{r}_u \times \vec{r}_v = \langle 2u, -1, 2v\rangle.$$

To decide on $\pm\vec{r}_u \times \vec{r}_v$, pick $u=v=0 \Rightarrow \vec{r}_u \times \vec{r}_v = \langle 0,-1,0\rangle$ - points outwards

$\Rightarrow$ we keep $\vec{r}_u \times \vec{r}_v$.

Hence: $\iint_{S_1} \vec{F}\,d\vec{S} = \iint_{u^2+v^2\le 1} \langle 0, u^2+v^2, -v\rangle \cdot \langle 2u, -1, 2v\rangle\,dA = \int_0^{2\pi}\int_0^1 (-r^2\cos^2\theta - 3r^2\sin^2\theta)\cdot r\,dr\,d\theta$

After straightforward computations (using $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$, $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$), we get $\iint_{S_1} \vec{F}\,d\vec{S} = -\pi$. Therefore: $\iint_S \vec{F}\,d\vec{S} = \pi - \pi = 0$ ∎