LECTURE #21

* Last time: Flux of $\vec{F}$ across S
  - Key step: decide on $\pm$ sign in front of $\vec{r}_u \times \vec{r}_v$

Ex 1: Do exercise Ex3 from Lecture #20 notes.

* This Week: Stokes' Theorem ← 3D analogue of Green's Theorem, relating line integral over the boundary of surface S and a flux across S.

Definition/Convention: The orientation of a surface S (given by a unit normal vector at all points) induces the positive orientation of the boundary curve $C = \partial S$. This means that if you walk in this "positive direction" around C with your head pointing in the direction of $\vec{n}$ then S is always on the left.

Rmk 1: If S is actually a region in xy-plane and $\vec{n} = \vec{k}$ at all points of S, then this definition of ∂S being positively oriented coincides with our old definition which appeared in the formulation of Green's Theorem.

Stokes' Theorem: Let S be an oriented piecewise smooth surface bounded by a simple closed piecewise smooth curve $C = \partial S$, endowed with a positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partials on an open region containing S. Then:

$$\boxed{\int_C \vec{F}\, d\vec{r} = \iint_S \operatorname{curl}(\vec{F})\, dS}$$

We will use this result in 2 ways:

1) reducing a computation of a line integral to a surface integral, which will often be easier to compute. Here we may choose any surface S whose boundary is C!
2) reducing a surface integral of a curl-vector field (which is tested by verifying if div is zero or not) to a line integral ⟶ either uncurl the field / ⟶ or change the surface keeping same boundary

Rmk 2: Recall that $\vec{G}$ is a curl of some vector field if and only if $\operatorname{div}(\vec{G}) = 0$.

For 1): Step 1 → Pick any surface S whose boundary is C
↑ always try the simplest one

Step 2 → Compute $\operatorname{curl}(\vec{F})$

Step 3 → Determine orientation of S and compute $\iint_S \operatorname{curl}(\vec{F})\, dS$

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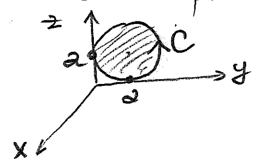
Ex2: Let C be a curve defined by a parametric equation $\vec{r}(t) = \langle 0, 2+2\cos t, 2+2\sin t\rangle$, $0 \le t \le 2\pi$

Evaluate $\int_C x^2 e^{5z}\,dx + x\cos y\,dy + 3y\,dz$.

! It is a good exercise to evaluate this line in a straightforward way: reduce to $\int_0^{2\pi} \left(12\cos t + 12\cos^2 t\right) dt$

- Draw the picture



C – circle of radius 2, centered at $(0,2,2)$, in yz-plane

C is oriented as shown

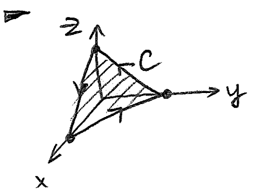
- The simplest surface S whose boundary is C is the disk bounded by C.
- At each point of S: $\vec{n} = \pm\hat{\imath}$ and to get orientation compatible with that of C we must pick $\vec{n} = \hat{\imath}$.
- $\mathrm{curl}(\vec{F}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2e^{5z} & x\cos y & 3y \end{vmatrix} = 3\hat{\imath} + 5x^2e^{5z}\hat{\jmath} + \cos y\,\hat{k} \Rightarrow \mathrm{curl}(\vec{F})\cdot\hat{\imath} = 3$

So: $\int_C x^2e^{5z}dx + x\cos y\,dy + 3y\,dz \overset{\text{Stokes}}{=} \iint_S \langle 3, 5x^2e^{5z}, \cos y\rangle\, d\vec{S} = \iint_S \langle 3, 5x^2e^{5z}, \cos y\rangle \cdot \hat{\imath}\, dS =$

$= \iint_S 3\,dS = 3A(S) = 3\cdot 4\pi = \boxed{12\pi}$ ∎

Ex3: Evaluate $\int_C \vec{F}\,d\vec{r}$, where $\vec{F}(x,y,z) = \langle z^2, y^2, x\rangle$ and C is the triangle with vertices $(1,0,0), (0,1,0), (0,1,0)$ walked over in this direction.

! This line integral can be computed in a straightforward way – try at home!



- The simplest surface S, whose boundary is C, is the interior of this Δ.
- We can parametrize it via $\vec{r}(u,v) = \langle u, v, 1-u-v\rangle$, $(u,v)\in D = \left\{\begin{matrix}(u,v): \\ u,v\ge 0 \\ u+v\le 1\end{matrix}\right\}$

  $\overset{\text{general formula}}{\Longrightarrow} \vec{r}_u \times \vec{r}_v = \langle 1,1,1\rangle$.
- It is clear that we must take $+\vec{r}_u\times\vec{r}_v$ to get compatible orientation
- $\mathrm{curl}(\vec{F}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ z^2 & y^2 & x \end{vmatrix} = (2z-1)\hat{\jmath}$

So: $\int_C \vec{F}\,d\vec{r} \overset{\text{Stokes}}{=} \iint_S \mathrm{curl}(\vec{F})\,d\vec{S} = \iint_D \langle 0, \underbrace{2z-1}_{2(1-u-v)-1}, 0\rangle\cdot\langle 1,1,1\rangle\,dA = \int_0^1\int_0^{1-u}(1-2u-2v)\,dv\,du$

$= \int_0^1 \left[(1-2u)v - v^2\right]\Big|_0^{v=1-u} du = \int_0^1\left((1-2u)(1-u)-(1-u)^2\right)du = \int_0^1 (u^2-u)\,du = \left(\frac{u^3}{3}-\frac{u^2}{2}\right)\Big|_{u=0}^{u=1} = \boxed{-\frac{1}{6}}$ ∎

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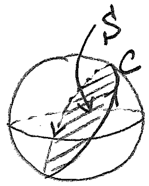
Ex 4: Evaluate $\int_C \vec{F}\,d\vec{r}$, where $\vec{F}(x,y,z) = \langle z-y, -x-z, -x-y\rangle$ and C is the curve $x^2+y^2+z^2=4$, $y=z$, oriented counterclockwise when viewed from above



- $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ z-y & -x-z & -x-y \end{vmatrix} = 2\hat{\jmath}$
- Simplest choice for S is the disk enclosed by C - in the plane $y=z$.

  So: $\int_C \vec{F}\,d\vec{r} = \iint_S 2\hat{\jmath}\,dS$

"Cheap" solution

As S is in the plane $y=z \Rightarrow$ unit normal vector at each point is $\pm\frac{1}{\sqrt{2}}(0,1,-1)$. Looking at the picture, we see that it is actually $(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$. But then:
$2\hat{\jmath}\cdot\hat{n} = \langle 0,2,0\rangle\cdot\langle 0,-1/\sqrt{2}, 1/\sqrt{2}\rangle = -\sqrt{2}$, so that

$$\iint_S 2\hat{\jmath}\,dS = \iint_S -\sqrt{2}\,dS = -\sqrt{2}\,A(S) = -\sqrt{2}\cdot 2^2\pi = \boxed{-4\sqrt{2}\,\pi}$$

"Honest/Canonical solution"

Parametrize S via $\vec{r}(u,v) = \langle u,v,v\rangle$, with $(u,v)\in D = \{(u,v) \mid u^2+2v^2\le 4\}$ - ellipse

$\Rightarrow \vec{r}_u\times\vec{r}_v = \langle 1,0,0\rangle\times\langle 0,1,1\rangle = \langle 0,-1,1\rangle$ and we must take it with "+" sign (explain!)

So: $\iint_S 2\hat{\jmath}\,dS = \iint_D \langle 0,2,0\rangle\cdot\langle 0,-1,1\rangle\,dA = -2\underbrace{A(D)}_{\pi\cdot 2\cdot\sqrt{2}} = \boxed{-4\sqrt{2}\,\pi}$

Ex 5: Evaluate $\int_C \vec{F}\,d\vec{r}$, $\vec{F}(x,y,z) = \langle y^2, 2x^2, e^{z^2}\rangle$ and C is the curve of intersection of the hemisphere $\begin{cases} x^2+y^2+z^2=25 \\ z\ge 0\end{cases}$ and the cylinder $x^2+y^2=9$, oriented counterclockwise when viewed from above.

- C is just the circle $\{(x,y,4) \mid x^2+y^2=9\}$
- Hence, the simplest S is the disk enclosed by C, parametrized via $\vec{r}(u,v)=\langle u,v,4\rangle$ with $D=\{(u,v) \mid u^2+v^2\le 9\}$.
- Also clear that $\vec{n} = \pm\hat{k}$ on S and we <u>must</u> choose $+\hat{k}$ so that orientations agree.
- $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 & 2x^2 & e^{z^2}\end{vmatrix} = (4x-2y)\hat{k}$

So: $\int_C \vec{F}\,d\vec{r} \overset{\text{Stokes}}{=} \iint_S (4x-2y)\hat{k}\,dS = \iint_S (4x-2y)\hat{k}\cdot\hat{k}\,dS = \iint_S (4x-2y)\,dS =$

$= \iint_D (4u-2v)\,dA = \int_0^{2\pi}\int_0^3 (4r\cos\theta - 2r\sin\theta)\,r\,dr\,d\theta = \boxed{0}$

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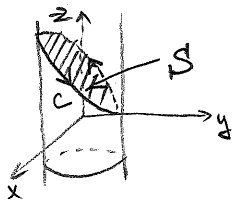
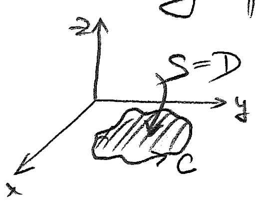
Ex6: Evaluate $\int_C \vec{F}\,d\vec{r}$, where $\vec{F}=\langle y^2,-x,-z^2\rangle$ and C is the curve of intersection of the plane $2y+z=1$ and the cylinder $x^2+y^2=1$, oriented counterclockwise when viewed from above.



- The simplest choice of S is the interior of this ellipse C. It may be parametrized via
$$\vec{r}(u,v)=\langle u, v, 1-2v\rangle \text{ with domain } D=\{(u,v)\mid u^2+v^2\le 1\}$$
so that
$$\vec{r}_u\times\vec{r}_v=\langle -\partial_u(1-2v), -\partial_v(1-2v), 1\rangle=\langle 0,2,1\rangle$$
- Again, looking at the picture it is clear we should pick $+\vec{r}_u\times\vec{r}_v$ (not $-\vec{r}_u\times\vec{r}_v$).
- $$\mathrm{curl}(\vec{F})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \partial_x & \partial_y & \partial_z\\ y^2 & -x & -z^2\end{vmatrix}=(-2y-1)\vec{k}$$

So: $$\int_C \vec{F}\,d\vec{r}=\iint_S \mathrm{curl}(\vec{F})\,dS=\iint_S\langle 0,0,-2y-1\rangle\,dS=\iint_D\langle 0,0,-2v-1\rangle\cdot\langle 0,2,1\rangle\,dA$$
$$=\iint_D(-2v-1)\,dA \overset{\text{Polar}}{=\!=}\int_0^{2\pi}\int_0^1(-2r\sin\theta-1)\,r\,dr\,d\theta=\boxed{-\pi}$$

Rmk 3: In the case when C is a closed curve in xy-plane, Stokes' = Green's Thm. Indeed, given $\vec{F}=\langle P,Q,R\rangle$ and picking surface S to be the interior of C in xy-plane, we see that $\vec{n}=\vec{k}$, so that Stokes' Thm yields:



$$\int_C \vec{F}\,d\vec{r}=\iint_S \mathrm{curl}(\vec{F})\,dS=\iint_S \mathrm{curl}(\vec{F})\cdot\vec{k}\,dS=\iint_S(Q_x-P_y)\,dS$$

But parametrizing S via $x=u$, $y=v$, $z=0$, we see $S=D$, $dS=dA$, and we get $\iint_S(Q_x-P_y)\,dS=\iint_D(Q_x-P_y)\,dA$.

Also as $\int_C \vec{F}\,d\vec{r}$ is clearly independent of R, we arrive at the Green's Thm:

$$\boxed{\int_C P\,dx+Q\,dy=\iint_D(Q_x-P_y)\,dA}$$