LECTURE #22

11/14/2019

* Last time: Stokes' Theorem

$$\boxed{\int_C \vec{F}d\vec{r} = \iint_S \operatorname{curl}(\vec{F})dS}$$

- C – boundary of surface S
- orientations of C and S are "compatible"!

Question: What is the boundary of the following surfaces:

(a) S_1 – part of the cylinder $x^2+y^2=1$ with $0 \le z \le 1$

(b) $S_1 \cup S_2$ – union of S_1 from (a) and S_2: $x^2+y^2 \le 1, z=0$

(c) $S_1 \cup S_2 \cup S_3$ – union of S_1, S_2 from (b) and S_3: $x^2+y^2 \le 1$, $z=1$.

! Last time we used Stokes' Theorem to evaluate line integrals over closed curves in $\mathbb{R}^3$

this is analogous to Green's Theorem for computation of line integrals $\int_C \vec{F}d\vec{r}$, where C was a closed path in xy-plane.

Our Strategy:
1) Pick the "simplest" surface S whose boundary is C
2) Determine orientation of S compatible with the given orientation of C.
3) Compute $\operatorname{curl}(\vec{F})$
4) Finally, find the flux of $\operatorname{curl}(\vec{F})$ across S.

Ex1: See Ex5 from last time.

Ex2: See Ex6 from last time.

! Time permitted, go over the final Remark from LECTURE #21.

* Today: Using Stokes' Theorem for the computation of fluxes.

Recall: For conservative $\vec{F} = \nabla f$, we had two methods of computing $\int_C \vec{F}d\vec{r}$:
1) Find potential f and use FTLI
2) Change the path C and compute the line integral by hand.

Question: Given $\iint_S \vec{G}dS$, in which cases Stokes' Theorem can be applied for its evaluation?

Recall: $\vec{G}$ is a curl vector field (i.e. $\vec{G} = \operatorname{curl}(\vec{F})$ for some $\vec{F}$) if and only if $\boxed{\operatorname{div}(\vec{G})=0}$

Moral: Verify if $\operatorname{div}(\vec{G})=0$ $\xrightarrow{NO}$ Stokes' theorem does not apply

$\xrightarrow{YES}$ Can apply Stokes' Theorem

Either $\vec{G}$ is already given as $\operatorname{curl}(\vec{F})$, in which case you just find $\int_C \vec{F}d\vec{r}$

As we didn't discuss "uncurling" $\vec{G}$, if $\operatorname{div}\vec{G}=0$ but $\vec{G}$ is not explicitly given as $\operatorname{curl}(\vec{F})$ CHANGE the surface!

$\iint_S \vec{G}dS = \iint_{S'} \vec{G}dS$, where $\partial S = \partial S'$ and the orientations of S, S' agree!

LECTURE #22

Ex3: If $\vec{F}(x,y,z) = \langle \cos^3(y), \sin(z^4), x^2 \rangle$ and S is the hemisphere $\begin{cases} x^2+y^2+z^2=9 \\ z \geq 0 \end{cases}$ oriented outward, evaluate $\iint_S \vec{F}\, d\vec{S}$.

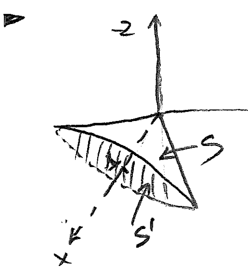
- $\operatorname{div}(\vec{F}) = 0+0+0=0 \Rightarrow \vec{F}$ is a curl vector field, but clearly it's hard to uncurl it.
- Replace S by S': $x^2+y^2 \leq 9$, $z=0$ (draw the picture!)
- The compatible orientation of S' is given by $\vec{n} = +\hat{k}$ at each point (explain this!)

So: $\iint_S \vec{F}\, d\vec{S} = \iint_{S'} \vec{F}\, d\vec{S} = \iint_{S'} \vec{F}\cdot\vec{n}\, dS = \iint_{S'} \vec{F}(x,y,z)\cdot\hat{k}\, dS = \iint_{S'} x^2\, dS \;⊜$

Parametrize S': $\vec{r}(u,v) = \langle u, v, 0\rangle$ with domain $D = \{(u,v) \mid u^2+v^2 \leq 9\}$

$$⊜ \iint_D u^2\, dA \;\underset{v=r\sin\theta}{\overset{u=r\cos\theta}{=\!=}}\; \int_0^{2\pi}\!\!\int_0^3 r^2\cos^2\theta\, r\, dr\, d\theta = \int_0^3 r^3 dr \cdot \int_0^{2\pi} \underbrace{\cos^2\theta}_{\frac{1+\cos(2\theta)}{2}}\, d\theta = \boxed{\frac{81}{4}\pi}$$

Ex4: If $\vec{F} = \langle z^3, -3x, y^2 \rangle$ and S is the part of $x = \sqrt{y^2+z^2}$ with $0 \leq x \leq 1$, oriented outward, evaluate $\iint_S \vec{F}\, d\vec{S}$



- $\operatorname{div}(\vec{F}) = 0+0+0=0 \Rightarrow$ Stokes' Applies
- Replace S by S': $y^2+z^2 \leq 1$, $x=1$
- The compatible orientation of S' is given by $\vec{n} = -\hat{\imath}$ (explain this!)

So: $\iint_S \vec{F}\, d\vec{S} = \iint_{S'} \vec{F}\, d\vec{S} = \iint_{S'} \vec{F}\cdot(-\hat{\imath})\, dS = -\iint_{S'} z^3\, dS \;⊜$

Parametrize S': $\vec{r}(u,v) = \langle 1, u, v\rangle$ with domain $D = \{(u,v) \mid u^2+v^2 \leq 1\}$

$$⊜ \iint_D (-v^3)\, dA \;\underset{v=r\sin\theta}{\overset{u=r\cos\theta}{=\!=}}\; -\int_0^{2\pi}\!\!\int_0^1 r^3\sin^3\theta \cdot r\, dr\, d\theta = -\frac{1}{5}\int_0^{2\pi} \sin^3\theta\, d\theta =$$

$$= -\frac{1}{5}\int_0^{2\pi} \sin^2\theta \cdot \sin\theta\, d\theta = -\frac{1}{5}\int_0^{2\pi} (1-\cos^2\theta)\cdot d(\cos\theta)\cdot(-1) = \boxed{0}$$

Question: What is $\iint_S \operatorname{curl}(\vec{F})\, d\vec{S}$ for any closed surface S? (it is ZERO!)
no boundary!

Ex5: Evaluate $\iint_S \operatorname{curl}(\vec{F})\, d\vec{S}$, where $\vec{F} = \langle z^2, -3xy, x^3y^3\rangle$ and S is the part of paraboloid $z = 5-x^2-y^2$ above the plane $z=1$, oriented upwards.

The boundary C of S is the circle $x^2+y^2=4$, $z=1$, oriented counterclockwise when viewed from above. Parametrize C via $\vec{r}(t) = \langle 2\cos t, 2\sin t, 1\rangle$ with t from 0 to 2π.

Stokes': $\iint_S \operatorname{curl}(\vec{F})\, d\vec{S} = \int_C \vec{F}\, d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\, dt = \int_0^{2\pi} \langle 1, -12\cos t\sin t, 64\cos^3 t\sin^3 t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\, dt$

$$= \int_0^{2\pi} (-2\sin t - 24\cos^2 t\sin t)\, dt = -24\int_0^{2\pi} \cos^2 t\sin t\, dt \;\overset{u=\cos t}{=\!=}\; 24\int_1^1 u^2\, du = \boxed{0}$$