* Last time : Flux integrals $\iint_S \vec{G}\,dS$ via Stokes' Theorem
  - applies only if $\operatorname{div}(\vec{G})=0$, which guarantees $\vec{G}=\operatorname{curl}(\vec{F})$ for some $\vec{F}$
  - either find $\vec{F}$ (in practice, we do not expect you to uncurl $\vec{G}$) and use

$$\iint_S \vec{G}\,dS = \int_{C=\partial S} \vec{F}\,d\vec{r}$$

  - or change the surface, i.e. pick another surface S' with the same boundary: $(\partial S = \partial S')$

$$\iint_S \vec{G}\,dS = \iint_{S'} \vec{G}\,dS$$

but you will need to evaluate the flux over S' directly

! The orientation of S' should be chosen so that the positive orientations of $\partial S=\partial S'$ induced from orientations of S and S', coincide.

In practice:

1) either determine first orientation of $\partial S=\partial S'$ from that of S, and then determine that of S'.
2) evoking that $\iint_{S\cup(-S')} \vec{G}\,dS=0$ as $S\cup\underbrace{\text{"}(-S')\text{"}}_{\text{surface } S' \text{ with opposite orientation to what we are looking}}$ has no boundary

Here the unit normal vector on $S\cup\text{"}-S\text{"}$ changes "continuously" ← illustrate on examples from last time.

Hence: in the equality $\iint_S \vec{G}\,dS = \iint_{S'} \vec{G}\,dS$, the orientation of S' should be opposite to the just determined.

Ex1: Do Ex5 from Lecture #22.

Final Remark: We have two "similar" theorems:

$$\boxed{\int_C \nabla f\,d\vec{r} = f(Q)-f(P)} \quad \text{and} \quad \boxed{\iint_S \operatorname{curl}(\vec{F})\,dS = \int_C \vec{F}\,d\vec{r}}$$

* Today: Triple integrals in Cartesian coordinates

Similar to $\int_a^b f(x)\,dx$ and $\iint_D f(x,y)\,dV$, one can view the triple integral of a continuous function $f(x,y,z)$ on a solid R in $\mathbb{R}^3_{xyz}$, denoted $\boxed{\iiint_R f(x,y,z)\,dV}$ as representing the "signed" 4-dimensional volume "under" the graph of $w=f(x,y,z)$ "over" the region R.

! As it is impossible to picture 4-dim object, we will just set bounds of triple integrals.

Simplest solid: Rectangular box $\boxed{B=[a,b]\times[c,d]\times[r,s]=\{(x,y,z)\,|\,a\le x\le b,\ c\le y\le d,\ r\le z\le s\}}$

Fubini's Thm: For f-continuous on B: $\boxed{\iiint_B f(x,y,z)\,dV = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz}$

! There are 6 possibilities to set up the iterated triple integrals, depending on the order of dx, dy, dz.

LECTURE #23

Ex 2: Evaluate $\iiint_B 6xye^z\,dV$, $B=[-1,2]\times[0,2]\times[0,1]$.

$$\iiint_B 6xye^z\,dV=\int_0^1\int_0^2\underbrace{\int_{-1}^2 6xye^z\,dx}_{3x^2ye^z\big|_{x=-1}^{x=2}=9ye^z}\,dy\,dz=\int_0^1\underbrace{\int_0^2 9ye^z\,dy}_{\frac{9}{2}y^2e^z\big|_{y=0}^{y=2}=18e^z}\,dz=\int_0^1 18e^z\,dz=18(e-1)$$

Rmk: In general, $\boxed{\iiint_{B=[a,b]\times[c,d]\times[r,s]} f(x)\cdot g(y)\cdot h(z)\,dV=\int_a^b f(x)dx\cdot\int_c^d g(y)dy\cdot\int_r^s h(z)dz}$

Similar to double integrals, given a general region $E\subseteq\mathbb{R}^3_{xyz}$ and a function $f(x,y,z)$ on E, define $\boxed{\iiint_E f(x,y,z)dV:=\iiint_B F(x,y,z)dV}$ with $E\subseteq B$ - rectangular box

$$F(x,y,z):=\begin{cases} f(x,y,z), & \text{if } (x,y,z)\in E\\ 0, & \text{if } (x,y,z)\in B\setminus E.\end{cases}$$

In practice, that leads to iterated integrals with non-constant bounds of integration.

Example 1: If $E=\{(x,y,z)\mid (y,z)\in D,\ u_1(y,z)\le x\le u_2(y,z)\}$, then

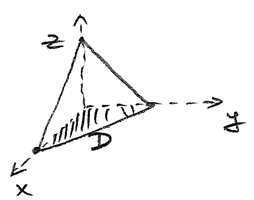
$$\boxed{\iiint_E f(x,y,z)dV=\iint_D\left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\right]dA}$$

← D is the projection of E onto yz-plane

Example 2: If $E=\{(x,y,z)\mid a\le x\le b,\ g_1(x)\le y\le g_2(x),\ u_1(x,y)\le z\le u_2(x,y)\}$, then

$$\boxed{\iiint_E f(x,y,z)dV=\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

Ex 3: Set up the triple integral of $f(x,y,z)=6xy$ over E - the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.



E.g. we can represent E as a collection of all points (x,y,z) with $0\le z\le 1-x-y$ and $(x,y)\in D=$ projection of E onto xy-plane, i.e. $D=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le 1-x\}$.

So: $\iiint_E 6xy\,dV=\int_0^1\int_0^{1-x}\int_0^{1-x-y} 6xy\,dz\,dy\,dx$ (actual answer is $\frac{1}{2}-1+\frac{3}{4}-\frac{1}{5}$)

Geometric / Physical Meaning

1) $\iiint_E 1\,dV=\text{Volume}(E)$

2) If $\rho(x,y,z)$ is the density function of a solid object occupying region E, then
$m=\iiint_E\rho(x,y,z)dV$ is the mass!

while

$\left(\frac{1}{m}\iiint_E x\rho(x,y,z)dV,\ \frac{1}{m}\iiint_E y\rho(x,y,z)dV,\ \frac{1}{m}\iiint_E z\rho(x,y,z)dV\right)$ - coordinates of the center of mass!

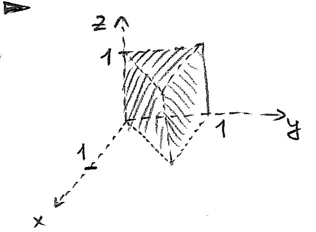
LECTURE #23

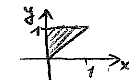
Ex 4: If $f(x,y,z)=x$ and R is the solid bounded by $z=1$, $z=0$, $y=x$, $y=1$, $x=0$, evaluate $\iiint_R f(x,y,z)\,dV$ with

a) dz first

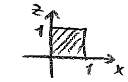
b) dy first

c) dx first

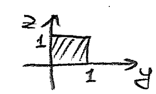
▶ 

a) $0\le z\le 1$ while $(x,y)\in D=\{(x,y)\mid 0\le x\le 1,\ x\le y\le 1\}$ 

$$\iiint_R x\,dV=\int_0^1\int_x^1\underbrace{\int_0^1 x\,dz}_{x}\,dy\,dx=\int_0^1\underbrace{\int_x^1 x\,dy}_{x-x^2}\,dx=\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_{x=0}^{x=1}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

b) $x\le y\le 1$ with $(x,z)\in D=\{(x,z)\mid 0\le x\le 1,\ 0\le z\le 1\}$ 

$$\iiint_R x\,dV=\int_0^1\int_0^1\underbrace{\int_x^1 x\,dy}_{x-x^2}\,dz\,dx=\int_0^1\underbrace{\int_0^1(x-x^2)\,dz}_{x-x^2}\,dx=\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_{x=0}^{x=1}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

c) $0\le x\le y$ with $(y,z)\in D=\{(y,z)\mid 0\le y\le 1,\ 0\le z\le 1\}$ 

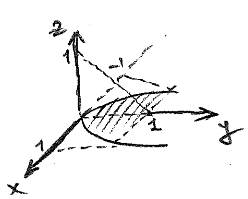
$$\iiint_R x\,dV=\int_0^1\int_0^1\underbrace{\int_0^y x\,dx}_{\frac{x^2}{2}\big|_{x=0}^{x=y}=\frac{y^2}{2}}\,dz\,dy=\int_0^1\underbrace{\int_0^1\frac{y^2}{2}\,dz}_{\frac{y^2}{2}}\,dy=\int_0^1\frac{y^2}{2}\,dy=\frac{y^3}{6}\Big|_{y=0}^{y=1}=\frac{1}{6}$$

■

Ex 5: Set up the triple integral of $f(x,y,z)=xy$ over the solid E that lies under the plane $z=2+x+y$ and above the region in the xy-plane bounded by $y=x^2$, $y=0$, $x=1$.

▶ $\displaystyle\int_0^1\int_0^{x^2}\int_0^{2+x+y} xy\,dz\,dy\,dx$ (actual answer is $\frac{1}{6}+\frac{1}{14}+\frac{1}{24}$) ■

Ex 6: Find the volume of the solid E enclosed by the cylinder $y=x^2$ and the planes $z=0$ and $y+z=1$.

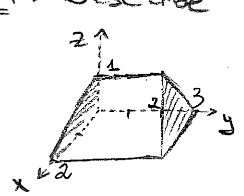
▶ 

$$Vol(E)=\iiint_E 1\,dV=\int_{-1}^1\int_{x^2}^1\underbrace{\int_0^{1-y}1\,dz}_{1-y}\,dy\,dx=\int_{-1}^1\underbrace{\int_{x^2}^1(1-y)\,dy}_{\left(y-\frac{y^2}{2}\right)\big|_{y=x^2}^{y=1}=\frac{1}{2}-x^2+\frac{x^4}{2}}\,dx=$$

$$=\int_{-1}^1\left(\frac{1}{2}-x^2+\frac{x^4}{2}\right)dx=\left(\frac{1}{2}x-\frac{1}{3}x^3+\frac{x^5}{10}\right)\Big|_{x=-1}^{x=1}=1-\frac{2}{3}+\frac{1}{5}$$ ■

Rmk: If there is a variable (x or y or z) which appears only in 2 of the governing equations, then place it for inner integral while the outside integration should be over the projection onto the remaining coordinate plane.

Ex 7: Describe the solid whose volume is given by $\displaystyle\int_0^2\int_0^{1-\frac{x}{2}}\int_0^{3-z}dy\,dz\,dx$.

▶ 

$0\le x\le 2$

$0\le z\le 1-\frac{x}{2}$

$0\le y\le 3-z$

■