LECTURE #24 11/21/2019

* Triple integrals in cylindrical coordinates

|| Def: The transformation $\vec{r}(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$ is called cylindrical coordinates

Rmk: It is unimportant which coordinate is left alone (you pick polar in the other two!).

Ex1: (a) Describe the surface whose equation in cylindrical coordinates is $z = 2r$

(b) —"— $z^2+r^2=9$.

► (a) $z=2r \Leftrightarrow z = 2\sqrt{x^2+y^2}$ – determines upper half of a cone

(b) $z^2+r^2=9 \Leftrightarrow z^2+x^2+y^2=9$ – determines a sphere of radius 3 centered at the origin ■

Theorem: If R is a region in Cartesian coordinates which corresponds to D in cylindrical coordinates, then

$$\boxed{\iiint_R f(x,y,z)\,dV = \iiint_D f(\vec{r}(r,\theta,z))\,\underset{\smile}{r}\,dz\,dr\,d\theta}$$

same factor as when switching to polar.
(just b/c we do polar change in x,y-coord.)

Ex2: Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$, where E is the region inside the cylinder $x^2+z^2=4$ and between the planes $y=-1$ and $y=2$.

► Here we leave y alone, while change to polar in x,z, i.e. cylindrical coord-s are r,θ,y.

$$\iiint_E \sqrt{x^2+z^2}\,dV = \int_0^{2\pi}\int_0^2\int_{-1}^2 r\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 3r^2\,dr\,d\theta = \int_0^{2\pi} 8\,d\theta = \boxed{16\pi}$$ ■

Ex3: Find the volume of the solid E enclosed by the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=2$.

► Their intersection is a circle $\{(x,y,1)\,|\,x^2+y^2=1\} \Rightarrow$ projection D of E onto xy-plane is unit disc.

$$Vol(E) = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r\sqrt{2-r^2}-r^2)\,dr\,d\theta = 2\pi\Big(\int_0^1 r\sqrt{2-r^2}\,dr - \underbrace{\frac{r^3}{3}\Big|_{r=0}^{r=1}}_{1/3}\Big)$$

u-substitution with $u=2-r^2 \Rightarrow du=-2r\,dr \Rightarrow \int_0^1 r\sqrt{2-r^2}\,dr = -\frac{1}{2}\int_2^1 u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_{u=1}^{u=2} = \frac{2\sqrt{2}-1}{3}$

So: $Vol(E) = \frac{4\pi(\sqrt{2}-1)}{3}$ ■

Ex 4: If R is the solid $0\le y\le x^2+z^2$ with $x^2+z^2\le 1$, set up $\iiint_R yz\,dV$.

► $$\int_0^{2\pi}\int_0^1\int_0^{r^2} y\cdot r\sin\theta\cdot r\,dz\,dr\,d\theta$$ ■

LECTURE #24

* Triple integrals in spherical coordinates

Let us recall that when we were computing integrals across the sphere $x^2+y^2+z^2=R^2$, it was convenient to parametrize it as follows:

$$\boxed{\vec{r}(\theta,\phi)=\langle R\cos\theta\sin\phi,\ R\sin\theta\sin\phi,\ R\cos\phi\rangle}$$

Def: The transformation

$$\vec{r}(\rho,\theta,\phi)=\langle \rho\cos(\theta)\sin(\phi),\ \rho\sin(\theta)\sin(\phi),\ \rho\cos(\phi)\rangle$$

is called spherical coordinates.

Ex 5: (a) The point $(\rho=3,\ \theta=\frac{\pi}{3},\ \phi=\frac{\pi}{4})$ is given in spherical coordinates. Find its Cartesian coord.

(b) Find spherical coordinates of the point $(3,0,4)$ in xyz-coordinates.

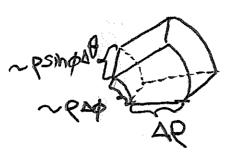
(a) $x=3\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{3\sqrt{2}}{4}$, $y=3\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{3\sqrt{6}}{4}$, $z=3\cdot\frac{\sqrt{2}}{2}=\frac{3\sqrt{2}}{2}$

(b) $\rho=\sqrt{3^2+0^2+4^2}=5$, $\theta=0$, $\phi=\cos^{-1}(\frac{4}{5})$ ∎

Theorem: If R is a region in Cartesian coordinates which corresponds to D in spherical coordinates, then

$$\boxed{\iiint_R f(x,y,z)\,dV=\iiint_D f(\vec{r}(\rho,\theta,\phi))\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi}$$

Rmk: To see the origin of this extra factor $\rho^2\sin\phi$, consider the "small wedge"



whose volume $\approx(\Delta\rho)\cdot(\rho\Delta\phi)\cdot(\rho\sin\phi\,\Delta\theta)=(\rho^2\sin\phi)\Delta\rho\Delta\theta\Delta\phi$

Also: the same factor $\rho^2\sin\phi$ already came up before when we were integrating over a sphere (ρ was replaced by radius R)!

Ex 6: Find the volume of a sphere of radius 1 using spherical coordinates

$$Vol(E)=\iiint_E 1\,dV=\int_0^{\pi}\int_0^{2\pi}\int_0^1\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\int_0^{\pi}\int_0^{2\pi}\frac{1}{3}\sin\phi\,d\theta\,d\phi=\int_0^{\pi}\frac{2\pi}{3}\sin\phi\,d\phi=\frac{4\pi}{3}$$ ∎

Ex 7: Find the volume of the solid that lies above the cone $z=\sqrt{\frac{x^2+y^2}{3}}$ and below the sphere $x^2+y^2+(z-\frac{1}{2})^2=1/4$

Equation of a cone: $\rho\cos\phi=\frac{\rho\sin\phi}{\sqrt{3}}\iff\tan\phi=\sqrt{3}\iff\boxed{\phi=\pi/3}$

Equation of a sphere: $x^2+y^2+z^2=z\iff\rho^2=\rho\cos\phi\iff\boxed{\rho=\cos\phi}$

$$\Rightarrow E=\left\{(\rho,\theta,\phi)\,\middle|\,\begin{array}{l}0\le\theta\le2\pi\\0\le\phi\le\pi/3\\0\le\rho\le\cos\phi\end{array}\right\}$$

So: $$Vol(E)=\iiint_E dV=\int_0^{\pi/3}\int_0^{2\pi}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\frac{2\pi}{3}\int_0^{\pi/3}\cos^3\phi\sin\phi\,d\phi$$

$$\underset{du=-\sin\phi d\phi}{\overset{u=\cos\phi}{=}}\ \frac{2\pi}{3}\int_1^{1/2}u^3(-du)=\frac{2\pi}{3}\cdot\frac{u^4}{4}\Big|_{u=1/2}^{u=1}=\boxed{\frac{\pi}{6}\left(1-1/16\right)}$$ ∎