LECTURE #25 12/03/2019

* Last time: triple integrals in cylindrical and spherical coordinates.

Ex1: (a) Set up the integral of $f(x,y,z) = e^{\frac{1}{3}(x^2+y^2+z^2)^{3/2}}$ over the ball $B = \{(x,y,z) \mid x^2+y^2+z^2 \le 4\}$ in Cartesian coordinates. Can you compute it?

(b) Same question, but in cylindrical coordinates.

(c) Same question, but in spherical coordinates.

▸ (a) $$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} e^{\frac{1}{3}(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx$$

(b) $$\int_{0}^{2\pi}\int_{0}^{2}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} e^{\frac{1}{3}(r^2+z^2)^{3/2}}\cdot r\,dz\,dr\,d\theta$$

(c) $$\int_0^{\pi}\int_0^{2\pi}\int_0^{2} e^{\frac{1}{3}\rho^3}\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\int_0^{2\pi}\sin(\phi)\cdot\left(e^{\frac{1}{3}\rho^3}\Big|_{\rho=0}^{\rho=2}\right)d\theta\,d\phi = 2\cdot 2\pi\,(e^{8/3}-1) = \boxed{4\pi(e^{8/3}-1)}$$ ■

Ex2: Set up the integrals evaluating $\iiint_B f(x,y,z)\,dV$.

(a) B – the "inside" part of the cone $z=\sqrt{\frac{x^2+y^2}{3}}$ bounded from above by the plane Π containing the curve of intersection of this cone and the sphere $x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}$.
Use both spherical and cylindrical coordinates.

(b) B – part of the ball $x^2+y^2+(z-\frac{1}{2})^2 \le \frac{1}{4}$ lying above the plane Π from (a).
Use both spherical and cylindrical coordinates.

(c) B – part of the ball $x^2+y^2+(z-\frac{1}{2})^2\le\frac{1}{4}$ which is outside the cone $z\ge\sqrt{\frac{x^2+y^2}{3}}$
Use both spherical and cylindrical coordinates.

▸ (a) The curve of intersection is given by

$$\begin{cases} x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4} \iff x^2+y^2+z^2=z \\ z=\sqrt{\frac{x^2+y^2}{3}} \iff x^2+y^2=3z^2 \ \&\ z\ge 0 \end{cases} \Rightarrow \quad 4z^2=z \Rightarrow z=\frac{1}{4} \ \left(\text{if } z=0 \Rightarrow x=y=0 \Rightarrow \text{get the origin which is not on } \Pi\right)$$

So: Π is given by $z=\frac{1}{4}$, while the curve of intersection is a circle of radius $\frac{\sqrt{3}}{4}$ on Π.

* Thus, in cylindrical coordinates we get

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}/4}\int_{r/\sqrt{3}}^{1/4} f(r\cos\theta, r\sin\theta, z)\cdot r\,dz\,dr\,d\theta}$$

* In spherical coordinates, we get (recall $z=\rho\cos(\phi) \Rightarrow \rho = z/\cos(\phi)$):

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_0^{\pi/3}\int_0^{2\pi}\int_0^{1/4\cos(\phi)} f(\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi}$$

(b) As for the points on the sphere, we get $\rho^2=x^2+y^2+z^2=z=\rho\cos(\phi) \Rightarrow \rho=\cos(\phi)$, we get

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_0^{\pi/3}\int_0^{2\pi}\int_{1/4\cos(\phi)}^{\cos(\phi)} f(\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))\,\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi}$$

In cylindrical coordinates z ranges from $\frac{1}{4}$ (only 4 $r\le\sqrt{3}/4$) up to larger solution of $r^2+(z-\frac{1}{2})^2=\frac{1}{4}$:

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}/4}\int_{1/4}^{\frac{1}{2}+\sqrt{\frac{1}{4}-r^2}} f(r\cos\theta, r\sin(\theta), z)\cdot r\,dz\,dr\,d\theta + \int_0^{2\pi}\int_{\sqrt{3}/4}^{1/2}\int_{\frac{1-\sqrt{1-4r^2}}{2}}^{\frac{1+\sqrt{1-4r^2}}{2}} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta}$$

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- (Continuation of Ex2)

(c) In cylindrical coordinates:

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}/4}\int_{\frac{1-\sqrt{1-4r^2}}{2}}^{1/4} f(r\cos\theta, r\sin\theta, z)\cdot r\,dz\,dr\,d\theta}$$

In spherical coordinates:

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_{\pi/3}^{\pi/2}\int_0^{2\pi}\int_0^{\cos(\phi)} f(\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))\cdot \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi}$$

Ex3: Evaluate the integral (by changing to spherical coordinates)
(#15.8.43)

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{3/2}\,dz\,dy\,dx$$

- This equals $\iiint_E (x^2+y^2+z^2)^{3/2}\,dV$, where E is given by $\begin{cases} -2\le x\le 2 \\ -\sqrt{4-x^2}\le y\le\sqrt{4-x^2} \\ 2-\sqrt{4-x^2-y^2}\le z\le 2+\sqrt{4-x^2-y^2}\end{cases}$,

i.e. E is the ball $\{(x,y,z) \mid x^2+y^2+(z-2)^2\le 4\}$.

Let us rewrite it in spherical coordinates:

- $0\le\theta\le 2\pi$
- $0\le\phi\le\pi/2$
- equation of the sphere reads $\rho^2 = 4z = 4\rho\cos\phi \Rightarrow \rho = 4\cos(\phi)$

So: get $\int_0^{\pi/2}\int_0^{2\pi}\int_0^{4\cos(\phi)} \rho^3\cdot\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\int_0^{2\pi}\left(\frac{\rho^6}{6}\sin(\phi)\right)\Big|_{\rho=0}^{\rho=4\cos(\phi)} d\theta\,d\phi =$

$= 2\pi\cdot\frac{4^6}{6}\cdot\int_0^{\pi/2}\cos^6(\phi)\sin(\phi)\,d\phi \xrightarrow[du=-\sin(\phi)d\phi]{u=\cos(\phi)} \frac{4^6}{3}\pi\int_1^0 u^6(-du) = \frac{4^6}{3\cdot 7}\pi = \boxed{\frac{4096}{21}\pi}$

Ex4: Let R be portion with $z\ge 0$ of the solid $x^2+y^2+z^2\le 4$ outside the cylinder $x^2+y^2=1$.

Set up $\iiint_R (y+z)\,dV$ in cylindrical and spherical coordinates.

- In cylindrical coordinates $0\le\theta\le 2\pi$, $1\le r\le 2$, $0\le z\le\sqrt{4-r^2}$, so we get

$$\boxed{\iiint_R (y+z)\,dV = \int_0^{2\pi}\int_1^2\int_0^{\sqrt{4-r^2}} (r\sin\theta+z)\cdot r\,dz\,dr\,d\theta}$$

In spherical coordinates $0\le\theta\le 2\pi$, $\frac{\pi}{6}\le\phi\le\frac{\pi}{2}$ (as the curve of intersection is $\begin{cases}x^2+y^2=1\\ z=\sqrt{3}\end{cases}$ and $z=\rho\cos(\phi)\Rightarrow\cos(\phi)=\frac{\sqrt{3}}{2}\Rightarrow\phi=\pi/6$) while the upper bound for ρ is clearly 2, and the lower bound is determined via $1 = x^2+y^2 = \rho^2\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}\sin^2(\phi) \Rightarrow \rho = \frac{1}{\sin(\phi)}$

So:
$$\boxed{\iiint_R (y+z)\,dV = \int_{\pi/6}^{\pi/2}\int_0^{2\pi}\int_{1/\sin(\phi)}^{2} (\rho\sin(\theta)\sin(\phi)+\rho\cos(\phi))\cdot\rho^2\sin(\phi)\cdot d\rho\,d\theta\,d\phi}$$

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* Today: Divergence Theorem

Theorem: Let E be a bounded closed solid region in $\mathbb{R}^3$ with a piece-wise smooth boundary S endowed with an outward orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region containing E. Then:

$$\boxed{\iint_S \vec{F}\,d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\,dV}$$

Ex 5: Find the flux of $\vec{F} = \langle y+e^{z^2}, -\sin(x^3)+z^{10}, 3z - x^{200}y^{100}\rangle$ over the sphere $S: x^2+y^2+z^2=4$, oriented inwards.

- $\operatorname{div}(\vec{F}) = 0+0+3 = 3$
- S is the boundary of the ball $B = \{(x,y,z) \mid x^2+y^2+z^2 \leq 4\}$.

So: $\iint_S \vec{F}\,d\vec{S} = \underset{\text{opposite orientation}}{-}\iiint_B \operatorname{div}(\vec{F})\,dV = -3\cdot \operatorname{Vol}(B) = -3\cdot\frac{4}{3}\pi\cdot 2^3 = \boxed{-32\pi}$

Ex6: If $\vec{F}$ is a curl-field and S is a closed surface (oriented), what is $\iint_S \vec{F}\,d\vec{S}$.

As $\operatorname{div}(\operatorname{curl}(\vec{F}))=0$, we get $\iint_S \vec{F}\,d\vec{S} = \pm 0 = 0$ by the divergence theorem

Rmk: This already follows from Stokes Thm.

Ex7: Evaluate $\iint_S \vec{F}\,d\vec{S}$, where $\vec{F} = \langle xy, -\frac{1}{2}y^2, z\rangle$ and the surface S (oriented outwards) consists of (1) $z = 4-3x^2-3y^2$ with $1\leq z\leq 4$, (2) $x^2+y^2=1$, $0\leq z\leq 1$, and (3) $x^2+y^2\leq 1$, $z=0$.

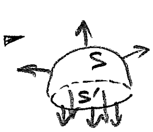
$\operatorname{div}(\vec{F}) = y-y+1 = 1$

In cylindrical coordinates, the solid E bounded by S is given by $\begin{cases} 0\leq\theta\leq 2\pi \\ 0\leq r\leq 1 \\ 0\leq z\leq 4-3r^2 \end{cases}$

So: $\iint_S \vec{F}\,d\vec{S} \underset{\text{Div. Thm}}{=\!=} \iiint_E 1\,dV = \int_0^{2\pi}\int_0^1\int_0^{4-3r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (4r-3r^3)\,dr\,d\theta = 2\pi\left(2r^2 - \frac{3}{4}r^4\right)\Big|_{r=0}^{r=1}$

$= 2\pi\cdot\frac{5}{4} = \boxed{\frac{5\pi}{2}}$

Ex8: Evaluate $\iint_S \vec{F}\,d\vec{S}$, where $\vec{F} = \langle x+e^z, y+\sin(z^2), 2z+1\rangle$ and S is the hemisphere $\begin{cases} x^2+y^2+z^2=1 \\ z\geq 0\end{cases}$ oriented upwards.



To apply Div. Thm, we "close S" by adding the disc S': $x^2+y^2\leq 1$, $z=0$. (oriented downwards)

Then: $\iint_S \vec{F}\,d\vec{S} + \iint_{S'} \vec{F}\,d\vec{S} \underset{\text{Div. Thm}}{=\!=} \iiint_E \operatorname{div}(\vec{F})\,dV = 4\operatorname{Vol}(E) = 4\cdot\frac{4\pi}{3}\cdot\frac{1}{2} = \frac{8\pi}{3}$

But: $\iint_{S'} \vec{F}\,d\vec{S} = \iint_{S'} \vec{F}\cdot(-\vec{k})\,dS = \iint_{S'}(-1)\,dA = -\operatorname{Area}(S') = -\pi$

$\Rightarrow \boxed{\iint_S \vec{F}\,d\vec{S} = \frac{11\pi}{3}}$

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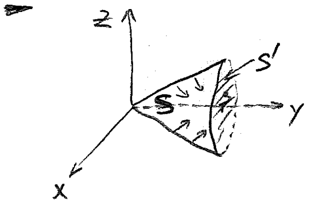
Ex9: Let $\vec{F} = \langle e^{y^2}+z,\ 2y^2,\ 3\sin(xy)+2\rangle$ and let S be the cone $y=\sqrt{x^2+z^2}$ for $y\le 3$, oriented in the positive y-direction. Evaluate $\iint_S \vec{F}\,d\vec{S}$.



$$\operatorname{div}(\vec{F}) = 0+4y+0 = 4y$$

$$\iint_{S\cup S' \text{ (oriented inwards)}} \vec{F}\,d\vec{S} = -\iiint_E \operatorname{div}(\vec{F})\,dV = -\int_0^{2\pi}\int_0^3 \underbrace{\int_r^3 4y\cdot r\,dy}_{2ry^2\big|_{y=r}^{y=3} = 18r-2r^3}\,dr\,d\theta$$

E: solid bounded by $S\cup S'$

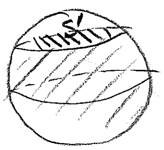
$$= -\int_0^{2\pi} \left(9r^2-\tfrac{1}{2}r^4\right)\Big|_{r=0}^{r=3} d\theta = -2\pi\left(81-\tfrac{81}{2}\right) = -81\pi$$

But: $\iint_{S'} \vec{F}\,d\vec{S} = \iint_{S'} \vec{F}\cdot(-\vec{j})\,dS = \iint_{S'} -18\,dS = -18\,\text{Area}(S') = -18\cdot 9\pi$ $\Rightarrow$

$$\Rightarrow \boxed{\iint_S \vec{F}\,d\vec{S} = 81\pi}$$

Ex10: Let $\vec{F} = \langle \sin(x^2)+y,\ 2x+3y^2,\ e^{z^2}-y\rangle$ and S be the portion of the sphere $x^2+y^2+z^2=4$ with $z\le 1$, oriented outwards. Evaluate $\iint_S \operatorname{curl}(\vec{F})\,d\vec{S}$.

- $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \sin(x^2)+y & 2x+3y^2 & e^{z^2}-y \end{vmatrix} = \vec{i}\cdot(-1) - \vec{j}\cdot 0 + \vec{k}\cdot(2-1) = \langle -1,0,1\rangle.$



Approach 1: One can compute this flux in a straightforward way, where S is parametrized via (ϕ,θ) with $0\le\theta\le 2\pi$, $\frac{\pi}{3}\le\phi\le\pi$ ← explain this!

$\vec{r}(\phi,\theta) = \langle 2\sin(\phi)\cos(\theta),\ 2\sin(\phi)\sin(\theta),\ 2\cos(\phi)\rangle$

$\vec{r}_\phi\times\vec{r}_\theta \overset{\text{Lecture 19}}{=} \langle 4\sin^2(\phi)\cos(\theta),\ 4\sin^2(\phi)\sin(\theta),\ 4\sin(\phi)\cos(\phi)\rangle$ ← correct orientation

So: $$\iint_S \operatorname{curl}(\vec{F})\,d\vec{S} = \int_{\pi/3}^{\pi} \underbrace{\int_0^{2\pi} \underbrace{\langle -1,0,1\rangle\cdot\langle 4\sin^2(\phi)\cos(\theta),\ 4\sin^2(\phi)\sin(\theta),\ 4\sin(\phi)\cos(\phi)\rangle}_{4\sin(\phi)\cos(\phi)-4\sin^2(\phi)\cos(\theta)}\,d\theta}_{8\pi\cdot\sin(\phi)\cos(\phi)}\,d\phi$$

$$= \int_{\pi/3}^{\pi} 8\pi\sin(\phi)\cos(\phi)\,d\phi \ \overset{\substack{u=\sin(\phi)\\ du=\cos(\phi)d\phi}}{=} \int_{\frac{\sqrt{3}}{2}}^{0} 8\pi t\,dt = 4\pi t^2\Big|_{t=\frac{\sqrt{3}}{2}}^{t=0} = \boxed{-3\pi}$$

Approach 2: Apply Stokes theorem. But due to $\sin(x^2)$-term, we will not be able to evaluate the line integral $\int_C \vec{F}\,d\vec{r}$.

Instead: replace S by S', oriented downwards (explain why)

Then: $\iint_S \operatorname{curl}(\vec{F})\,d\vec{S} = \iint_{S'} \operatorname{curl}(\vec{F})\,d\vec{S} = \iint_{S'} \langle -1,0,1\rangle\cdot\langle 0,0,-1\rangle\,dS = -\text{Area}(S') = \boxed{-3\pi}$

Approach 3: Apply divergence theorem to $S\cup S'$ with S' oriented upwards to find

$$\iint_{S\cup S'} \operatorname{curl}(\vec{F})\,d\vec{S} = 0 \Rightarrow \iint_S \operatorname{curl}(\vec{F})\,d\vec{S} = -\iint_{S' \text{(upward orient)}} \operatorname{curl}(\vec{F})\,d\vec{S} \overset{\text{Approach 2}}{=} \boxed{-3\pi}$$