Lecture 26: Review

Objectives:

1. Know all of the last two thirds of the course.

1 Review

You have learned 6 different types of integrals (yes, there are seven different notations listed):

$$\int_C \mathbf{F} \cdot d\mathbf{r}, \quad \iint_D f(x,y)dA, \quad \int_C F_1 dx + F_2 dy, \quad \iint_S f(x,y,z)dS,$$

$$\iiint_R f(x,y,z)dV, \quad \iint_S \mathbf{F} \cdot d\mathbf{S}, \quad \int_C f ds.$$

For each of the following, identify the notation for the integral from the above list, and write in words what each represents geometrically.

- Scalar line integral:
- Double integral over xy-regions:
- Scalar Surface integrals:
- Triple Integrals over 3D solids:
- Vector Line Integrals (this has two possible notations):
- Vector Surface Integral:

Now, for each of the scalar integrals, describe how to compute them (or what choices you have to make):

- Scalar line integral:

- Double integral over xy-regions:

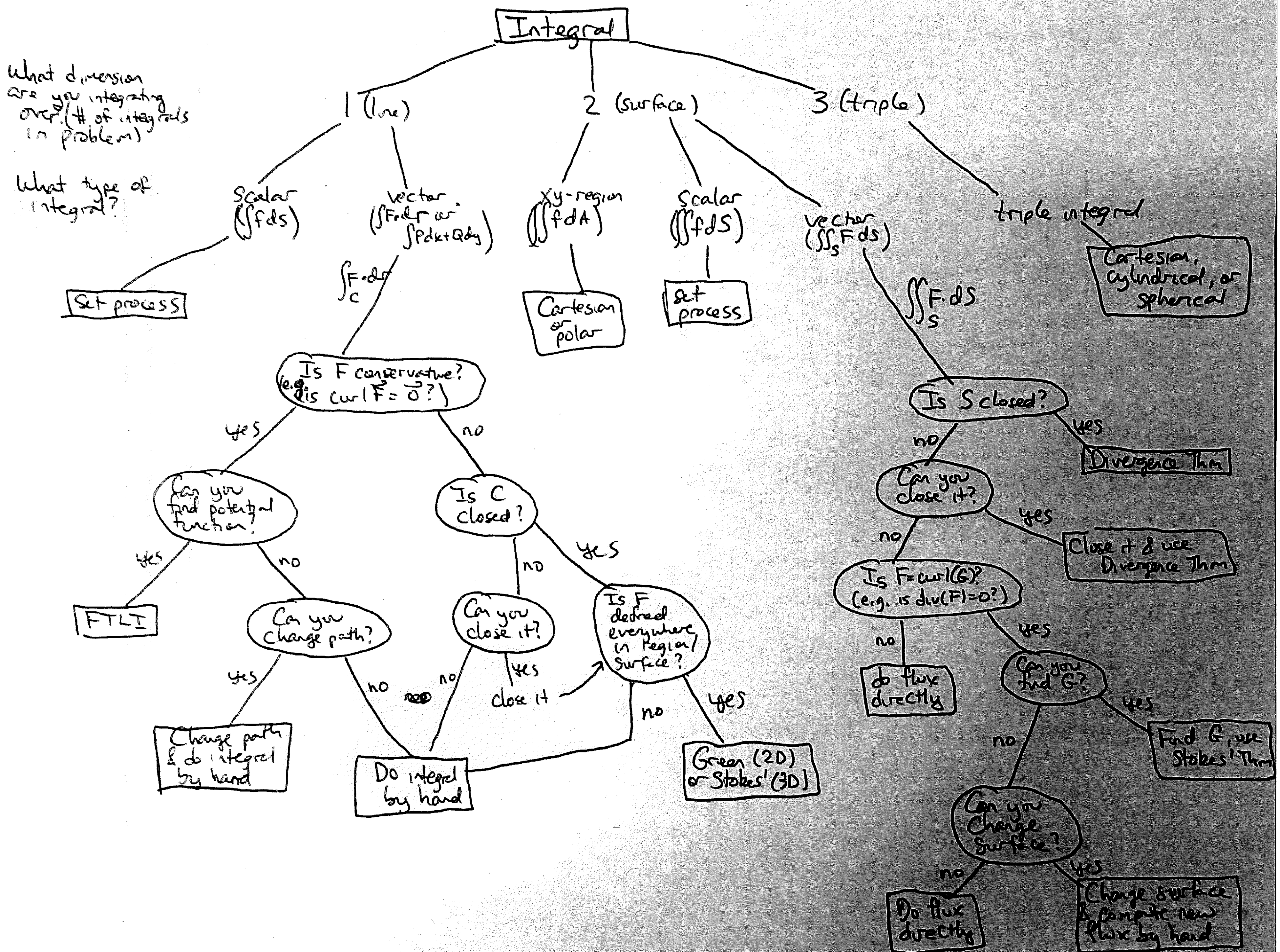
- Scalar Surface integrals:

- Triple Integrals over 3D solids:

Lastly, for each of the vector integrals, describe all the techniques/theorems that you can use to compute the integral, and what you have to check for each one.

- Vector Line Integrals (this has two possible notations):

- Vector Surface Integral:



What dimension are you integrating over? (# of integrals in problem)
What type of integral?
Integral
1 (line)
2 (surface)
3 (triple)
Scalar ($\int f\,ds$)
Set process
vector ($\int F\cdot dr$ or $\int P\,dx + Q\,dy$)
$\int_C F\cdot ds$
Is F conservative? (e.g. is curl $\vec{F} = \vec{0}$?)
yes
no
Can you find potential function?
FTLI
Can you change path?
Change path & do integral by hand
Do integral by hand
Is C closed?
Can you close it?
close it
Is F defined everywhere in region/surface?
Green (2D) or Stokes' (3D)
xy-region ($\iint f\,dA$)
Cartesian or polar
Scalar ($\iint f\,dS$)
Set process
vector ($\iint_S F\,ds$)
$\iint_S F\cdot ds$
Is S closed?
Divergence Thm
Can you close it?
Close it & use Divergence Thm
Is F = curl(G)? (e.g. is div(F) = 0?)
do flux directly
Can you find G?
Find G, use Stokes' Thm
Can you change surface?
Do flux directly
Change surface & compute new flux by hand
triple integral
Cartesian, cylindrical, or spherical

$$
\begin{array}{ccccccc}
\{\text{functions}\} & \xrightarrow{\nabla} & \{\text{vector fields}\} & \xrightarrow{\text{curl}} & \{\text{vector fields}\} & \xrightarrow{\text{div}} & \{\text{functions}\} \\
\Big\downarrow{\scriptstyle \text{plug in points}} & & \Big\downarrow{\scriptstyle \text{line integral}} & & \Big\downarrow{\scriptstyle \text{flux integral}} & & \Big\downarrow{\scriptstyle \text{triple integral}} \\
\{\text{points}\} & \xleftarrow[\text{boundary}]{} & \{\text{curves}\} & \xleftarrow[\text{boundary}]{} & \{\text{surfaces}\} & \xleftarrow[\text{boundary}]{} & \{\text{solids}\}
\end{array}
$$

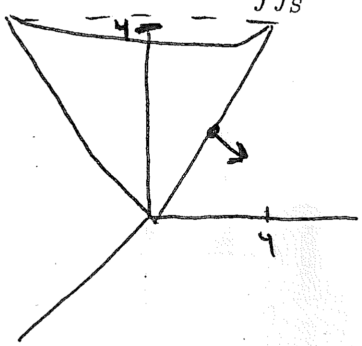
$$f(Q) - f(P) = \int_C \mathbf{F} \cdot d\mathbf{r} \qquad \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_R \operatorname{div}(\mathbf{F}) dV$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$$

Remember:

- To push forward, you are checking that the domain of integration is closed (either a closed curve or a closed surface).
- To move backwards, you are checking something about the vector field.
- To see whether you can push a vector field backwards, move forwards and check that you get 0 (either the zero vector or the zero function).
- Always check orientation.

Example 1. Let $\mathbf{F} = \langle e^z + 2y^2, 3x + 2y, x + y + z \rangle$, and let S be the portion of $z = \sqrt{x^2 + y^2}$ which lies inside the cylinder $x^2 + y^2 = 16$, oriented with outward normals. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.



$\text{div} F = 0 + 2 + 1 = 3$

Div Thm + Closing

Let S' be the disc $x^2+y^2 \le 16$ in plane $z=4$, oriented upward.

$$\iint_S F \cdot dS + \iint_{S'} F \cdot dS = \iiint_E 3\, dV$$

- $\iiint_E 3\, dV = 3 \text{ vol}(E)$ or $\int_0^{2\pi}\int_0^4\int_r^4 3r\, dz\, dr\, d\theta = 64\pi$

- $\iint_{S'} F \cdot dS$: $\vec{r}(x,y) = \langle x, y, 4 \rangle$ $R: x^2+y^2 \le 16$

  $r_x = \langle 1, 0, 0 \rangle$

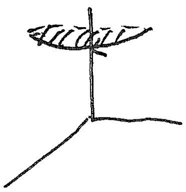
  $r_y = \langle 0, 1, 0 \rangle$

  $r_x \times r_y = \langle 0, 0, 1 \rangle$

  $$\iint_R (x+y+4)\, dA = \int_0^{2\pi}\int_0^4 (r\cos\theta + r\sin\theta + 4)\, r\, dr\, d\theta = 64\pi$$

$\therefore \iint_S F \cdot dS = 0.$

Example 2. Let C be the circle $x^2 + y^2 = 16$ in the plane $z = 4$, oriented clockwise when viewed from above, and let $\mathbf{F} = \langle e^x + 2y^2, 3x + 2y, x + y + z \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.



$$\text{curl}\, F = \langle 1, 1, 3 - 4y \rangle$$

$$\int_C F \cdot dr = \iint_S \text{curl}\, F \cdot dS$$

$S: z = 4,\ R: x^2 + y^2 \leq 16$ <u>downward</u>

$\vec{r}(x,y) = \langle x, y, 4 \rangle$

$r_x \times r_y = \langle 0, 0, 1 \rangle \to$ wrong normal

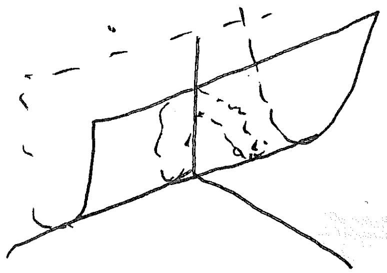
$-(r_x \times r_y) = \langle 0, 0, -1 \rangle$

$$\iint_R \langle 1, 1, 3 - 4y \rangle \cdot \langle 0, 0, -1 \rangle\, dA$$

$$= \iint_R (4y - 3)\, dA$$

$$= \int_0^{2\pi} \int_0^4 (4r\sin\theta - 3)\, r\, dr\, d\theta = \boxed{-48\pi}$$

Group Work 1. If $\mathbf{F} = \langle x^2 + 3zx, y^3 + x, \cos(z^2) + 2x \rangle$ and C is the intersection of $z = y^2$ with the cylinder $x^2 + y^2 = 1$, oriented counterclockwise when viewed from above, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.



curl F= $\langle 0, -2+3x, 1 \rangle$.

$\int_C F \cdot dr = \iint_S \text{curl} F \cdot dS$

S: $z = y^2$ w/ $x^2+y^2 \leq 1$ upward normals

$\vec{r}(x,y) = \langle x, y, y^2 \rangle$

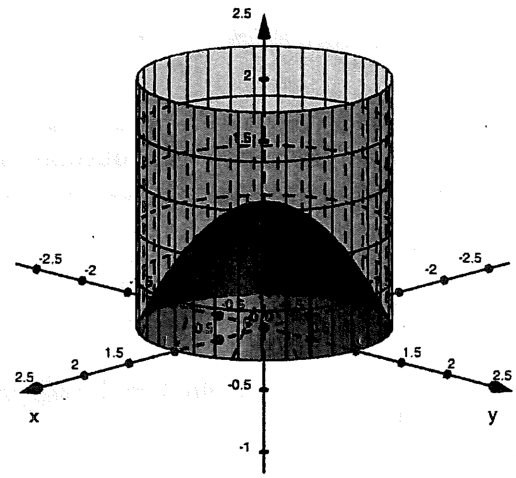
$r_x = \langle 1, 0, 0 \rangle$

$r_y = \langle 0, 1, 2y \rangle$

$r_x \times r_y = \langle 0, -2y, 1 \rangle$ correct.

$$\iint_{x^2+y^2 \leq 1} (4y - 6xy + 1)\, dA = \boxed{\pi}$$

Group Work 2. Let S be the surface consisting of the paraboloid $z = 1 - x^2 - y^2$ for $z \geq 0$, oriented downward, and the cylinder $x^2 + y^2 = 1$ for $0 \leq z \leq 2$ with outward normals, as shown below:



Let $\mathbf{F} = \langle x + 3z, y + z, z^2 - 10 \rangle$. Evaluate:

(a) $\iint_S \mathbf{F} \cdot d\mathbf{S}$

Answer. Here, S is not closed. As $\text{div}(\mathbf{F}) = 2 + 2z \neq 0$, we know $\mathbf{F} \neq \text{curl}(\mathbf{G})$, so we can either do the flux directly, or close the surface and use divergence theorem. We will do the latter (the flux would require two pieces). Let S' be the disc $x^2 + y^2 \leq 1$ in the plane $z = 2$, oriented upward. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} + \iint_{S'} \mathbf{F} \cdot d\mathbf{S} = \iiint_R (2 + 2z) dV,$$

where R is the region enclosed by S and S'. The triple integral can be done in cylindrical coordinates:

$$\iiint_R (2 + 2z) dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^2 (2 + 2z) r dz\, dr\, d\theta = \frac{20\pi}{3}.$$

The flux integral should be done directly. Parametrize S' as $\mathbf{G}(x, y) = \langle x, y, 2 \rangle$ in domain $D : x^2 + y^2 \leq 1$. Then $\mathbf{G}_x \times \mathbf{G}_y = \langle 0, 0, 1 \rangle$, which is the desired normal. Then

$$\begin{aligned} \iint_{S'} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \langle x + 6, y + 2, -6 \rangle \cdot \langle 0, 0, 1 \rangle \, dA \\ &= \iint_D -6 dA \\ &= -6\pi. \end{aligned}$$

Therefore,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{20\pi}{3} + 6\pi = \frac{38\pi}{3}.$$

(b) $\iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$

Answer. We can use Stokes' theorem to say

$$\iint_S \text{curl}(\mathbf{F}) d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Here the boundary C is the circle $x^2 + y^2 = 1$ in the plane $z = 2$. We need it to be clockwise when viewed from above (because of the outward normals of the cylinder). Parametrize the circle as $\mathbf{r}(t) = \langle \cos(t), \sin(t), 2 \rangle$ for$0 \leq t \leq 2\pi$. But this goes the wrong way. Therefore

$$\begin{aligned} \iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= -\int_0^{2\pi} \langle \cos(t) + 6, \sin(t) + 2, -6 \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle \, dt \\ &= 0. \end{aligned}$$

Try doing this problem with surface independence or divergence theorem (after closing) instead!

(c) If C is the boundary of S, given a positive orientation, evaluate $\int_C \text{curl}(\mathbf{F}) \cdot d\mathbf{r}$.

Answer. Notice $\text{curl}(\mathbf{F}) = \langle -1, 3, 0 \rangle$, and $\text{curl}(\text{curl}(\mathbf{F})) = \langle 0, 0, 0 \rangle$, so $\text{curl}(\mathbf{F})$ is conservative. Since C is closed, the line integral is 0.