HOMEWORK 2

1. Let $V \neq 0$ be a representation of an algebra A. We say that a vector $v \in V$ is **cyclic** if it generates V, i.e. $Av = V$. A representation admitting a cyclic vector is called **cyclic**.

Prove:

(a) V is irreducible if and only if all nonzero vectors are cyclic.

(b) V is cyclic is and only if it is isomorphic to A/I, where I is a left ideal of A.

(c) Give an example of an indecomposable representation which is not cyclic.

Hint: Let $A = \mathbb{C}[x, y]/I$, where I denotes the ideal spanned by homogeneous polynomials of degree ≥ 2. Let $V = A^*$ be the space of linear functionals on A, with the action of A given by $(\rho(a)f)(b) = f(ba)$. Prove that V provides such an example.

2. Let A be the Weyl algebra, that is $A = k\langle x, y\rangle/\langle yx - xy - 1\rangle$.

(a) If $\mathrm{char}(k) = 0$, what are the 2-sided ideals in A?

Hint: First, show that any nonzero 2-sided ideal contains a nonzero polynomial in x.

(b) Assume $\mathrm{char}(k) = p > 0$ (as always k is algebraically closed). Find the center of A.

(c) Assume $\mathrm{char}(k) = p > 0$ (as always k is algebraically closed). Find all irreducible finite dimensional representations of A.

Hint: First prove that all of them are p-dimensional.

3. The commutant $K(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the linear span of elements $[x, y],\ x, y \in \mathfrak{g}$. This is an ideal in $\mathfrak{g}$ (i.e. a subrepresentation of the adjoint representation). A finite dimensional Lie algebra $\mathfrak{g}$ over a field k is called **solvable** if there exists n such that $K^n(\mathfrak{g}) = 0$.

Prove *Lie theorem:* if $k = \mathbb{C}$ and V is a finite dimensional irreducible representation of a solvable Lie algebra $\mathfrak{g}$, then V is 1-dimensional.

Hint: Prove the result by induction in dimension.