HOMEWORK 3

1. (a) Show that for a finite quiver Q, the path algebra P_Q is generated by $\{p_i\}_{i\in I} \cup \{a_h\}_{h\in E}$ with the following defining relations:

- $\circ$ $\sum_{i\in I} p_i = 1$,
- $\circ$ $p_i p_j = \delta_{ij} p_i$,
- $\circ$ $a_h p_{h'} = a_h, \ a_h p_j = 0$ if $j \neq h'$,
- $\circ$ $p_{h''} a_h = a_h, \ p_j a_h = 0$ if $j \neq h''$,

where h', h'' denote the "outcoming" and "incoming" vertices of the edge $h \in E$ (as in the class).

(b) Verify that the constructions from the class establish bijections between isomorphism classes of representations of the path algebra P_Q and of the quiver Q.

(c) Justify what the notions of "irreducible" and "indecomposable" representations of Q should mean, so that they exactly correspond to the "irreducible" and "indecomposable" representations of the path algebra P_Q.

2. Let $A = \mathrm{Mat}_d(k)$. Prove the following two results we used in the class:

(a) $A \simeq A^{\mathrm{op}}$ as algebras.

(b) $A \simeq A^*$ as A-representations.

3. Let $A_1, A_2, \ldots, A_n$ be n algebras with units $1_1, 1_2, \ldots, 1_n$, respectively. Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$. Clearly $1_i 1_j = \delta_{ij} 1_i$ and $1 = 1_1 + 1_2 + \ldots + 1_n$ is the unit of A.

(a) For every representation of V of A, verify that $1_i V$ is a representation of A_i. Vice-verse, if $V_1, \ldots, V_n$ are representations of $A_1, \ldots, A_n$, verify that $V_1 \oplus \cdots \oplus V_n$ becomes naturally an A-representation.

(b) Show that a representation V of A is irreducible if and only if $1_i V$ is an irreducible nonzero representation of A_i for exactly one $1 \leq i \leq n$ and $1_j V = 0$ for $j \neq i$. Deduce the classification of irreducible A-representations in terms of those of A_i.

(c) Verify in a direct way that the only irreducible representation of $\mathrm{Mat}_d(k)$ is k^d and that every finite dimensional representation of $\mathrm{Mat}_d(k)$ is a direct sum of copies of k^d.

(d) Deduce an alternative proof of the classification result of finite dimensional representations of the algebra $\bigoplus_{i=1}^r \mathrm{Mat}_{d_i}(k)$ we had in the class.

4. Given an algebra A and two representations V, W of A, we would like to classify all representations U of A such that V is a subrepresentation of U and $U/V \simeq W$.

Suppose we have a representation U as above. As a vector space it can be (nonuniquely) identified with $V \oplus W$, so that for any $a \in A$ the corresponding operator $\rho_U(a)$ has a block triangular form

$$\rho_U(a) = \begin{bmatrix} \rho_V(a) & f(a) \\ 0 & \rho_W(a) \end{bmatrix},$$

where $f\colon A \to \mathrm{Hom}_k(W, V)$ is a linear map.

(a) What is the necessary and sufficient condition on $f(a)$ under which $\rho_U(a)$ is a representation? Maps f satisfying this condition are called **1-cocycles**. They form a vector space denoted by $Z^1(W, V)$.

(b) Let $X\colon W \to V$ be a linear map. The coboundary of X, denoted dX, is defined to be the function $A \to \mathrm{Hom}_k(W, V)$ given by $dX(a) = \rho_V(a)X - X\rho_W(a)$. Show that dX is a cocycle which vanishes if and only if X is a homomorphism of representations. Thus, coboundaries form a subspace $B^1(W, V) \subset Z^1(W, V)$, which is isomorphic to $\mathrm{Hom}_k(W, V)/\mathrm{Hom}_A(W, V)$. The quotient $Z^1(W, V)/B^1(W, V)$ is denoted by $\mathrm{Ext}^1(W, V)$.

(c) Show that if $f, f' \in Z^1(W, V)$ and $f - f' \in B^1(W, V)$, then the corresponding extensions U, U' are isomorphic representations of A. Conversely, if $\phi\colon U \to U'$ is an isomorphism such that

$$\phi = \begin{bmatrix} 1_V & \star \\ 0 & 1_W \end{bmatrix},$$

then $f - f' \in B^1(W, V)$. Thus the space $\mathrm{Ext}^1(W, V)$ "classifies" extensions of W by V.

(d) Assume that W, V are finite-dimensional irreducible representations of A. For any $f \in \mathrm{Ext}^1(W, V)$, let U_f be the corresponding extension. Show that U_f is isomorphic to $U_{f'}$ as representations if and only if f and f' are proportional. Thus, isomorphism classes (as representations) of nontrivial extensions of W by V (i.e. those not isomorphic to $W \oplus V$) are parametrized by the projective space $\mathbb{P}\mathrm{Ext}^1(W, V)$. In particular, any extension is trivial if and only if $\mathrm{Ext}^1(W, V) = 0$.

5. (a) Let $A = \mathbb{C}[x_1, \ldots, x_n]$, and V_a, V_b be the 1-dimensional A-representations in which the elements x_i act by a_i, b_i ($a_i, b_i \in \mathbb{C}$), respectively. Find $\mathrm{Ext}^1(V_a, V_b)$. Classify all 2-dimensional A-representations.

(b) Let B be the algebra over $\mathbb{C}$ generated by $x_1, \ldots, x_n$ with the defining relations $x_i x_j = 0$ for all i, j. Show that for $n > 1$ the algebra B has infinitely many nonisomorphic indecomposable representations.

(c) Let Q be a quiver without oriented cycles, and let P_Q be the path algebra of Q. Find irreducible representations of P_Q and compute Ext^1 between them. Classify all 2-dimensional representations of P_Q.