HOMEWORK 4

1. (a) Given three $\mathbf{k}$-vector spaces V, W, U, construct a natural bijection between $\mathbf{k}$-bilinear maps $V \times W \to U$ and $\mathbf{k}$-linear maps $V \otimes W \to U$.

(b) Prove that if $\{v_i\}$ is a basis of V, $\{w_j\}$ is a basis of W, then $\{v_i \otimes w_j\}$ is a basis of $V \otimes W$.

(c) Construct a natural isomorphism $V^* \otimes W \to \mathrm{Hom}(V, W)$ in case V is finite dimensional.

(d) Let V be a $\mathbf{k}$-vector space. Define the **n-th symmetric power of** V, denoted by $S^n V$, to be the quotient of $V^{\otimes n}$ by the subspace spanned by $T - s(T)$ where $T \in V^{\otimes n}$ and s is a transposition between two copies of V in $V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$. Likewise, define the **n-th exterior power of** V, denoted by $\Lambda^n V$, to be the quotient of $V^{\otimes n}$ by the subspace spanned by T such that $T = s(T)$ for some transposition s.

If $\{v_i\}$ is a basis of V, construct bases of $S^n V$ and $\Lambda^n V$.

If $\dim(V) = m$, find $\dim(S^n V)$ and $\dim(\Lambda^n V)$.

(e) Assuming $\mathrm{char}(\mathbf{k}) = 0$, find a natural identification between $S^n V$ and the subspace $\{T \in V^{\otimes n} | T = s(T) \text{ for all } s\}$, find a natural identification between $\Lambda^n V$ and the subspace $\{T \in V^{\otimes n} | T = -s(T) \text{ for all } s\}$.

(f) Let $A\colon V \to W$ be a $\mathbf{k}$-linear map. Construct natural $\mathbf{k}$-linear maps $A^{\otimes n}\colon V^{\otimes n} \to W^{\otimes n}$, $S^n A\colon S^n V \to S^n W$, $\Lambda^n A\colon \Lambda^n V \to \Lambda^n W$.

(g) In the setup of (f), assume that $V = W$ and that A has eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$. Find $\mathrm{Tr}(S^n A), \mathrm{Tr}(\Lambda^n A)$. Show that $\Lambda^m A = \det(A)\mathrm{Id}$. Deduce $\det(AB) = \det(A)\det(B)$.

(h) Define the **tensor algebra of** V, denoted by TV, to be $TV = \bigoplus_{n \geq 0} V^{\otimes n}$ with multiplication defined by $a \cdot b = a \otimes b$ for $a \in V^{\otimes r}, b \in V^{\otimes s}$. Verify that if $\dim(V) = m$, then TV is isomorphic to a free algebra in m generators.

(i) Define the **symmetric algebra of** V, denoted by SV, to be the quotient of TV by the ideal generated by $\{v \otimes w - w \otimes v | v, w \in V\}$. Verify that if $\dim(V) = m$, then SV is isomorphic to a polynomial algebra in m generators. Show that $SV = \bigoplus_{n \geq 0} S^n V$.

(j) Define the **exterior algebra of** V, denoted by ΛV, to be the quotient of TV by the ideal generated by $\{v \otimes v | v \in V\}$. Verify that if $\dim(V) = m$, then ΛV is isomorphic to the exterior algebra in m generators. Show that $\Lambda V = \bigoplus_{n \geq 0} \Lambda^n V$.

2. Let V and W be $\mathbf{k}$-vector spaces of dimensions m and n, respectively. Construct a natural algebra isomorphism $\mathrm{End}(V) \otimes \mathrm{End}(W) \to \mathrm{End}(V \otimes W)$. Note that choosing bases of V and W, this gives rise to an algebra isomorphism $\mathrm{Mat}_{m \times m}(\mathbf{k}) \otimes_{\mathbf{k}} \mathrm{Mat}_{n \times n}(\mathbf{k}) \to \mathrm{Mat}_{mn \times mn}(\mathbf{k})$. Verify that under this identification, the irreducible representation $\mathbf{k}^{mn}$ of $\mathrm{Mat}_{mn \times mn}(\mathbf{k})$ is isomorphic to the tensor product of irreducible representations $\mathbf{k}^m$ and $\mathbf{k}^n$ of $\mathrm{Mat}_{m \times m}(\mathbf{k})$ and $\mathrm{Mat}_{n \times n}(\mathbf{k})$, respectively.

3. (a) Let $\mathbf{k}$ be a field and K be an extension of $\mathbf{k}$. If A is an algebra over $\mathbf{k}$ show that $A \otimes_{\mathbf{k}} K$ is naturally an algebra over K. Likewise, if V is an A-module, then $V \otimes_{\mathbf{k}} K$ has a natural structure of a module over $A \otimes_{\mathbf{k}} K$.

(b) Show that if M, N are modules over a commutative algebra A, then $M \otimes_A N$ has a natural structure of an A-module.

4. Let A be the algebra of real-valued continuous functions on $\mathbb{R}$ which are periodic with period 1, i.e. $f(x+1) = f(x)$. Let M be the A-module of continuous functions f on $\mathbb{R}$ which are antiperiodic with period 1, i.e., $f(x+1) = -f(x)$.

(a) Show that A and M are indecomposable A-modules.

(b) Show that A is not isomorphic to M, but $A \oplus A$ is isomorphic to $M \oplus M$ as A-modules. *Therefore, Krull-Schmidt theorem fails for infinite dimensional modules.*