## HOMEWORK 5

1. Show that if the characteristic of the ground field $\mathbf{k}$ divides the order of the group $G$, then the number of isomorphism classes of irreducible representations of $G$ is strictly less than the number of conjugacy classes in $G$.

*Hint: Consider the element* $P = \sum_{g\in G} g \in \mathbf{k}[G]$*. Verify* $P^2 = 0$ *and deduce* $\mathrm{Tr}_V(P) = 0$ *for every finite dimensional* $G$*-representation* $V$*.*

2. Let $G$ be a finite group. Let $V_i$ be the irreducible complex representations of $G$. For every $i$, define

$$\psi_i := \frac{\dim(V_i)}{|G|} \sum_{g\in G} \chi_{V_i}(g) g^{-1} \in \mathbb{C}[G].$$

(a) Prove that $\psi_i$ acts as the identity on $V_i$ and as the null map on $V_j$ $(j \neq i)$.

(b) Prove that $\psi_i^2 = \psi_i$ and $\psi_i\psi_j = 0$ for $i \neq j$.

3. Let $V$ be a finite dimensional complex vector space, and let $GL(V)$ be the group of invertible endomorphisms of $V$. Verify that $S^n V$ and $\Lambda^m V$ $(m \leq \dim(V))$ are representations of $GL(V)$ in a natural way. Prove that they are irreducible.

4. Let $G$ be the group of symmetries of a regular $N$-gon (it has $2N$ elements).

(a) Describe all irreducible complex representations of this group (consider separately the cases of even and odd $N$).

(b) Let $V$ be the 2-dimensional complex representation of $G$ obtained by complexification of the standard representation on the real plane (the plane of the polygon). Find the decomposition of $V \otimes V$ in a direct sum of irreducible representations.