HOMEWORK 6

1. Decompose a tensor product of any two irreducible representations of S_4 into irreducibles.

2. Let $K \subset H \subset G$ be three groups, W–a representation of K, V–a representation of G.

(a) Show that the K-representations $\mathrm{Res}^H_K(\mathrm{Res}^G_H(V))$ and $\mathrm{Res}^G_K(V)$ are naturally isomorphic.

(b) Show that the G-representations $\mathrm{Ind}^G_H(\mathrm{Ind}^H_K(W))$ and $\mathrm{Ind}^G_K(W)$ are naturally isomorphic.

(c) Use Frobenius reciprocity to show that (a) is equivalent to (b).

3. Let $H \subset G$ be a pair of finite groups, let $\chi \colon H \to \mathbb{C}^\times$ be a homomorphism, and let $\mathbb{C}_\chi$ denote the corresponding 1-dimensional representation of H. Set

$$e_\chi := \frac{1}{|H|} \sum_{g \in H} \chi(g)^{-1} g \in \mathbb{C}[H].$$

Verify that it is an idempotent. Prove that G-representations $\mathrm{Ind}^G_H(\mathbb{C}_\chi)$ and $\mathbb{C}[G]e_\chi$ are naturally isomorphic (where G acts on the latter via left multiplication).

4. Consider S_3 as a subgroup of S_4 consisting of those permutations of the set $\{1, 2, 3, 4\}$ which fix 4. For every irreducible S_3-representation W, decompose $\mathrm{Ind}^{S_4}_{S_3}(W)$ into irreducibles.

5. During our proof of the fact that a dimension of any complex irreducible representation of a finite group G divides the order of G, we used the fact that any element of $\mathbb{Z}[G]$ satisfies a monic polynomial equation with integer coefficients. Prove this fact.

6. Let $\mathbf{k}[G]_1$ (respectively $\mathbf{k}[G]_2$) denote $\mathbf{k}[G]$ endowed with a natural $(\mathbf{k}[H], \mathbf{k}[G])$-bimodule (respectively $(\mathbf{k}[G], \mathbf{k}[H])$-bimodule) structure.

(a) Verify that $\mathrm{Res}^G_H(V)$, $\mathbf{k}[G]_1 \otimes_{\mathbf{k}[G]} V$, and $\mathrm{Hom}_{\mathbf{k}[G]}(\mathbf{k}[G]_2, V)$ are naturally isomorphic.

(b) Verify that $\mathrm{Ind}^G_H(W)$, $\mathbf{k}[G]_2 \otimes_{\mathbf{k}[H]} W$, and $\mathrm{Hom}_{\mathbf{k}[H]}(\mathbf{k}[G]_1, W)$ are naturally isomorphic.