HOMEWORK 7

1. Find the sum of dimensions of all irreducible representations of S_n.
Hint: Show that all of them are of real type and apply the Frobenius-Schur theorem.

2. For a Young diagram μ, let $A(\mu)$ be the set of Young diagrams obtained by adding a square to μ, and let $R(\mu)$ be the set of Young diagrams obtained by removing a square from μ.

(a) Prove $\mathrm{Res}_{S_{n-1}}^{S_n} V_\mu \simeq \bigoplus_{\lambda \in R(\mu)} V_\lambda$.

(b) Prove $\mathrm{Ind}_{S_{n-1}}^{S_n} V_\mu \simeq \bigoplus_{\lambda \in A(\mu)} V_\lambda$.

3. The content $c(\lambda)$ of a Young diagram λ is the sum $\sum_j \sum_{i=1}^{\lambda_j} (i-j)$. Let

$$C = \sum_{1 \le i < j \le n} (ij) \in \mathbb{C}[S_n]$$

be the sum of all transpositions. Prove that C acts on the Specht module V_λ by multiplication by $c(\lambda)$.

4. Let V be any finite dimensional representation of S_n. Prove that the element

$$E = (12) + (13) + \ldots + (1n) \in \mathbb{C}[S_n]$$

is diagonalizable and has integer eigenvalues on V which are between $1-n$ and $n-1$.