HOMEWORK 10

1. Complete the proof of the theorem we sketched out in the class, providing a complete classification of finite dimensional algebraic representations of $\mathrm{GL}(V)$ and a decomposition of $\mathbb{C}[\mathrm{GL}(V)]$ as a $\mathrm{GL}(V) \times \mathrm{GL}(V)$-module.

2. (a) Applying the classification result of the previous problem, deduce an explicit description of the irreducible $\mathfrak{sl}_2$-representations over $\mathbb{C}$.

(b) Provide an elementary proof of the classification of part (a) by looking at the generalized eigenvalues of h-action with the biggest real part.

(c) Use the central element (called the *Casimir element*) $C = ef + fe + \frac{h^2}{2}$ of the universal enveloping algebra $U(\mathfrak{sl}_2)$ to prove that any finite-dimensional representation of $\mathfrak{sl}_2$ is semisimple.

(d) For any two irreducible $\mathfrak{sl}_2$-representations V and W, find the decomposition of the representation $V \otimes W$ into irreducibles.

3. Let $R_{k,N}$ be the algebra of polynomials on the space of k-tuples of $N \times N$ matrices $(X_1, \ldots, X_k)$ invariant under simultaneous conjugation. An example of an element of $R_{k,N}$ is the function $T_w := \mathrm{Tr}(w(X_1, \ldots, X_k))$, where w is any finite word on a k-letter alphabet. Use the Schur-Weyl duality to prove that $R_{k,N}$ is generated by the elements T_w.