HOMEWORK 12 (TAKE-HOME FINAL EXAM)

1. Let H be an index 2 subgroup of G, and assume that a restriction of an irreducible G-representation V is not H-irreducible. As seen in the class: $\mathrm{Res}_H^G V \simeq W' \oplus W''$, where W', W'' are pairwise non-isomorphic irreducible H-representations. Show that W'' is conjugate to W'.

Hint: Verify that if both W', W'' were self-conjugate, then V would be reducible.

Recall the bijection between symmetric Young diagrams λ of size n and the split pairs (c', c'') of conjugacy classes in A_n. We use r to denote the number of hooks in λ, while $\{q_i\}_{i=1}^r$ will denote the cycle lengths of the corresponding c', c'' (that is, $q_i = 2\lambda_i - 2i + 1$ for $1 \le i \le r$).

Theorem: (a) If c', c'' is a pair of split conjugacy classes that do not correspond to λ, then

$$\chi'_\lambda(c') = \chi''_\lambda(c'') = \chi'_\lambda(c') = \chi''_\lambda(c'') = \frac{1}{2}\chi_\lambda(c' \cup c''),$$

where χ_λ denotes the character of the Specht S_n-module V_λ, while $\chi'_\lambda, \chi''_\lambda$ are the characters of the two irreducible A_n-representations whose direct sum is $\mathrm{Res}_{A_n}^{S_n} V_\lambda$.

(b) If c' and c'' correspond to λ, then

$$\chi'_\lambda(c') = \chi''_\lambda(c'') = x,\ \chi'_\lambda(c'') = \chi''_\lambda(c') = y,$$

with x and y being the following two numbers:

$$(\dagger) \qquad \frac{1}{2}\left((-1)^m \pm \sqrt{(-1)^m q_1 \cdots q_r}\right), \text{ where } m = \frac{1}{2}\sum_{i=1}^{r}(q_i - 1).$$

2. Prove this theorem following the arguments listed below.

Step 1. Let $q = (q_1 > \cdots > q_r)$ be a sequence of positive odd integers adding to n, and let $c' = c'(q), c'' = c''(q)$ be the corresponding pair of split conjugacy classes in A_n. Let λ be a self-conjugate partition of n. Assume that χ'_λ and χ''_λ take on the same values on each element of A_n that is not in c' or c''. Let $u = \chi'_\lambda(c') = \chi''_\lambda(c'')$ and $v = \chi'_\lambda(c'') = \chi''_\lambda(c')$.

(i) Show that u, v are real when m is even and $\overline{u} = v$ when m is odd.

(ii) Let $\vartheta = \chi'_\lambda - \chi''_\lambda$. Deduce from the equation $\langle \vartheta, \vartheta \rangle = 2$ that $|u - v|^2 = q_1 \cdots q_r$.

(iii) Show that λ is the partition that corresponds to q and that $u + v = (-1)^m$. Deduce that u, v are the numbers specified in $(\dagger)$.

Step 2. Prove the theorem by induction on n, and for a fixed n, look at that q which has the smallest q_1 and for which some character has values on the classes $c'(q)$ and $c''(q)$ other than those prescribed by the theorem.

(i) If $r = 1$, so that $q_1 = n = 2m + 1$, the corresponding self-conjugate partition is $\lambda = (m + 1, 1, \ldots, 1)$. By induction, Step 1 applies to χ'_λ and χ''_λ.

(ii) If $r > 1$, consider the embedding $H = A_{q_1} \times A_{n-q_1} \subset A_n = G$, and let X', X'' be the representations of G induced from the representations $W'_1 \boxtimes W'_2$ and $W''_1 \boxtimes W'_2$, where W'_1, W''_1 are the representations of A_{q_1} corresponding to q_1, W'_2 is one of the representations of A_{n-q_1} corresponding to $(q_2, \ldots, q_r)$, and $\boxtimes$ denotes the external tensor product. Show that X' and

X'' are conjugate representations of A_n, and their characters χ', χ'' take equal values on each pair of split conjugacy classes, with the exception of $c'(q), c''(q)$, and compute the values of these characters on $c'(q), c''(q)$.

(iii) Let $\vartheta = \chi' - \chi''$. Verify that $\langle \vartheta, \vartheta \rangle = 2$. Decomposing X', X'' into irreducibles, deduce that $X' \simeq Y \oplus W'_\lambda$, $X'' \simeq Y \oplus W''_\lambda$ for some self-conjugate representation Y and some self-conjugate partition λ of n.

(iv) Apply Step 1 to the characters χ'_λ and χ''_λ, and conclude the proof.

3. Compute the character tables of $\mathrm{PGL}_2(\mathbb{F}_q)$ and $\mathrm{PSL}_2(\mathbb{F}_q)$ (for odd q).

4. Verify that $q+4$ representations of $\mathrm{SL}_2(\mathbb{F}_q)$ we established in the last class are indeed irreducible, pairwise non-isomorphic, and exhaust a complete list of irreducible representations.

Recall that representations W', W'', X', X'' of those $q+4$ irreducible $\mathrm{SL}_2(\mathbb{F}_q)$-representations were constructed implicitly. To be precise, if $\tau\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ is the non-trivial character such that $\tau^2 = 1$, then we defined W', W'' as the two irreducible representations whose direct sum is $\mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_q)}_{\mathrm{SL}_2(\mathbb{F}_q)} V_{\tau,1}$. Likewise, for a character $\psi\colon \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ such that its restriction $\psi_C\colon C \to \mathbb{C}^\times$ (here $C = \{x \in \mathbb{F}_{q^2}^\times : x^{q+1} = 1\}$) is nontrivial but $\psi_C^2 = 1$, we defined X', X'' as the two irreducible representations whose direct sum is $\mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_q)}_{\mathrm{SL}_2(\mathbb{F}_q)} M_\psi$. The goal of the last problem is to determine the characters of these four representations.

5. Let H be the index 2 subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$ consisting of those matrices whose determinant is a square.

(a) Describe the split conjugacy classes in H. Verify that both restrictions $\mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_q)}_H V_{\tau,1}$ and $\mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_q)}_H M_\psi$ split into sums of conjugate irreducible representations of half their dimensions. Deduce that their further restrictions to $\mathrm{SL}_2(\mathbb{F}_q)$ remain irreducible and exactly recover W', W'', X', X''. Use this to prove that the character values of W', W'' (resp. X', X'') on all nonsplit conjugacy classes are the same as half the character of $V_{\tau,1}$ (resp. M_ψ).

(b) Let s, t, s', t' denote the remaining values of the characters of W', W'':

$$s = \chi_{W'}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \chi_{W''}\begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}, \quad t = \chi_{W'}\begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix} = \chi_{W''}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$s' = \chi_{W'}\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = \chi_{W''}\begin{pmatrix} -1 & \epsilon \\ 0 & -1 \end{pmatrix}, \quad t' = \chi_{W'}\begin{pmatrix} -1 & \epsilon \\ 0 & -1 \end{pmatrix} = \chi_{W''}\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Show that $s + t = 1$. Using $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and the equality $\chi(g^{-1}) = \overline{\chi(g)}$ for any character, deduce that s, t are real if $q \equiv 1 \pmod 4$ and $s = \bar{t}$ if $q \equiv 3 \pmod 4$. Use the equality $\chi(-g) = \chi(g) \cdot \chi(1)/\chi(-e)$ to prove $s' = \tau(-1)s, t' = \tau(-1)t$. Finally, use $\langle \chi_{W'}, \chi_{W'} \rangle = 1$ to determine $s\bar{t} + t\bar{s}$. Combining all the above, deduce that

$$s, t = \frac{1}{2} \pm \frac{1}{2}\sqrt{\omega q}, \text{ where } \omega = \tau(-1) = \begin{cases} 1 & \text{if } q \equiv 1 \pmod 4, \\ -1 & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

(c) Use analogous arguments to compute the characters of X', X''.