HOMEWORK 1 : COALGEBRAS AND BIALGEBRAS

1. Let $H = (H, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. Verify that

$$H^{\mathrm{op}} = (H, \mu^{\mathrm{op}}, \eta, \Delta, \epsilon),\ H^{\mathrm{cop}} = (H, \mu, \eta, \Delta^{\mathrm{op}}, \epsilon),\ H^{\mathrm{op,cop}} = (H, \mu^{\mathrm{op}}, \eta, \Delta^{\mathrm{op}}, \epsilon)$$

are also bialgebras (here $\Delta^{\mathrm{op}} := \tau_{H,H} \circ \Delta,\ \mu^{\mathrm{op}} := \mu \circ \tau_{H,H}$ with $\tau_{H,H}$ flipping the factors).

2. Let G be a finite group. We identify the vector space $\mathbf{k}(G)$ of $\mathbf{k}$-valued functions on G with $\mathbf{k}[G]$ via $\mathbf{k}(G) \ni f \mapsto \sum_{g \in G} f(g)g$. Write down explicit formulas for the unit, product, counit, and coprodcut on $\mathbf{k}(G)$, induced by the bialgebra structure on $\mathbf{k}[G]$ (which we discussed in the class). The resulting product on $\mathbf{k}(G)$ is called the *convolution product*.

3. (*An example of neither commutative nor cocommutative bialgebra*) Let H be the quotient of the free algebra $k\{x, t\}$ by the two-sided ideal generated by $t^2 - 1, x^2, xt + tx$. Prove that $\dim(H) = 4$ and that it has a natural bialgebra structure with

$$\Delta(x) = 1 \otimes x + x \otimes t,\ \Delta(t) = t \otimes t,\ \epsilon(x) = 0,\ \epsilon(t) = 1.$$

4. (a) Given two coalgebras (C, Δ, ϵ) and (C', Δ', ϵ'), verify that the linear maps

$$\pi \colon C \otimes C' \to C \text{ given by } c \otimes c' \mapsto \epsilon'(c')c \text{ and } \pi' \colon C \otimes C' \to C' \text{ given by } c \otimes c' \mapsto \epsilon(c)c'$$

are coalgebra morphisms.

(b) (*Universal property of the tensor product of coalgebras*) Prove that for any cocommutative coalgebra D and any pair of coalgebra morphisms $f \colon D \to C,\ f' \colon D \to C'$, there is a unique coalgebra morphism $f \otimes f' \colon D \to C \otimes C'$ such that $\pi \circ (f \otimes f') = f$ and $\pi' \circ (f \otimes f') = f'$.

5. (a) Given a vector space V, show that there is a unique bialgebra structure on the *tensor algebra* $T(V)$ such that $\Delta(v) = v \otimes 1 + 1 \otimes v$ and $\epsilon(v) = 0$ for any $v \in V$. Verify that this bialgebra is cocommutative. Prove the following formulas for any $v_1, \ldots, v_n \in V$ $(n > 0)$:

$$\epsilon(v_1 \ldots v_n) = 0,$$

$$\Delta(v_1 \ldots v_n) = v_1 \ldots v_n \otimes 1 + 1 \otimes v_1 \ldots v_n + \sum_{p=1}^{n-1} \sum_{\sigma-\text{shuffle}} v_{\sigma(1)} \ldots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \ldots v_{\sigma(n)},$$

where a permutation $\sigma \in S_n$ is *a $(p, n-p)$-shuffle* if $\sigma(1) < \ldots < \sigma(p),\ \sigma(p+1) < \ldots < \sigma(n)$.

(b) Recall the *symmetric algebra* $S(V)$, which is a quotient of $T(V)$ by the two-sided ideal generated by $\{xy - yx | x, y \in V\}$. Verify that the kernel $I := \mathrm{Ker}(T(V) \twoheadrightarrow S(V))$ is a coideal of $T(V)$. In particular, the bialgebra structure on $T(V)$ from (a) induces the one on $S(V)$.

(c) In the particular case of one-dimensional V, identify $S(V)$ with the polynomial algebra $k[t]$, so that $t^m \cdot t^n = t^{m+n}$ and $\Delta(t^n) = \sum_{k=0}^{n} \binom{n}{k} t^k \otimes t^{n-k}$.

(d) (*Divided powers*) If $\mathrm{char}(\mathbf{k}) = 0$ and $\dim(V) = 1$, find another identification (as vector spaces) of $S(V)$ and $k[t]$, such that $t^m \cdot t^n = \binom{m+n}{m} t^{m+n}$ and $\Delta(t^n) = \sum_{k=0}^{n} t^k \otimes t^{n-k}$.