HOMEWORK 2: HOPF ALGEBRAS AND COMODULES

1. (a) Given an algebra morphism $f\colon A \to A'$ and a coalgebra morphism $g\colon C \to C'$, consider the natural map $\mathrm{Hom}(C, A) \to \mathrm{Hom}(C', A')$ given by $h \mapsto f \circ h \circ g$. Prove that it is an algebra morphism with respect to the convolution $\star$.

(b) Let H be a Hopf algebra, A a commutative algebra, C a cocommutative coalgebra. Prove that the set of algebra morphisms $\mathrm{Hom}_{\mathrm{alg}}(H, A)$ (respectively, the set of coalgebra morphisms $\mathrm{Hom}_{\mathrm{coalg}}(C, H)$) is a group with respect to the convolution $\star$, the inverse of f being given by $f \circ S$ (respectively, by $S \circ f$).

(c) Prove that a bialgebra morphism between two Hopf algebras is necessarily a Hopf algebra morphism.

2. (a) Given a coalgebra (C, Δ, ϵ) and a vector space V, verify that $(C \otimes V, \Delta \otimes \mathrm{Id}_V)$ is a comodule. It is called a *free comodule*.

(b) Let (C, Δ, ϵ) be a coalgebra and (N, Δ_N) be a C-comodule. Prove that Δ_N is an injective comodule morphism from (N, Δ_N) to the free comodule $(C \otimes N, \Delta \otimes \mathrm{Id}_N)$.

(c) Let (C, Δ, ϵ) be a coalgebra and $(C \otimes V, \Delta \otimes \mathrm{Id}_V)$ be a free comodule. Show that for any comodule (N, Δ_N) the map $f \mapsto (\epsilon \otimes \mathrm{Id}_V) \circ f$ is a linear isomorphism from $\mathrm{Hom}_{\mathrm{comod}}(N, C \otimes V)$ to $\mathrm{Hom}_{\mathrm{vec}}(N, V)$.

3. Let $(H, \mu, \eta, \Delta, \epsilon, S)$ be a Hopf algebra.

(a) Define $\psi^0 := \eta\epsilon,\ \psi^n := \mathrm{Id}_H^{\star n},\ \psi^{-n} := S^{\star n}$ $(n \in \mathbb{Z}_{>0})$. Prove that each ψ^n is an algebra (respectively, coalgebra) endomorphism of H if the latter is commutative (respectively, cocommutative). In both cases verify $\psi^n \star \psi^m = \psi^{n+m}$ for any $n, m \in \mathbb{Z}$.

(b) For a group G, set $H := \mathbf{k}[G]$. Evaluate $\psi^n(g)$ for any $g \in G, n \in \mathbb{Z}$.

(c) Recall the Hopf algebra structure on $S(V)$ from [Homework 1, Problem 5(b)]. Evaluate $\psi^n(x)$ for any $x \in S^d(V), d \in \mathbb{N}, n \in \mathbb{Z}$.

4. An algebra A is graded if there exist subspaces $\{A_i\}_{i \in \mathbb{N}}$ such that $A = \oplus_{i \in \mathbb{N}} A_i$ and $A_i \cdot A_j \subset A_{i+j}$. Likewise, a coalgebra (C, Δ, ϵ) is graded if there exist subspaces $\{C_i\}_{i \in \mathbb{N}}$ such that $C = \oplus_{i \in \mathbb{N}} C_i$ and $\Delta(C_k) \subset \oplus_{i+j=k} C_i \otimes C_j,\ \epsilon(C_n) = 0$ for $n > 0$. Finally, given a graded vector space $V = \oplus_{i \in \mathbb{N}} V_i$, its graded dual is $V^*_{\mathrm{gr}} := \oplus_{i \in \mathbb{N}} (V_i)^*$.

(a) Prove that the graded dual of a graded coalgebra has a natural graded algebra structure.

(b) Prove that the graded dual of a graded algebra $A = \oplus A_i$ with finite-dimensional A_i has a natural graded coalgebra structure.

(c) Verify that the coalgebra structure on $\mathbf{k}[t]$ from [Homework 1, Problem 5(d)] is the graded dual of the usual algebra structure on $\mathbf{k}[t]$ (graded via $\deg(t) = 1$).