HOMEWORK 2, EXTRA PROBLEM 1

1. Bimodules

Let H be a bialgebra and M be a vector space equipped with an H-module and an H-comodule structures given by linear maps

$$\mu_M \colon H \otimes M \longrightarrow M \quad \text{and} \quad \Delta_M \colon M \to H \otimes M.$$

Endow $H \otimes M$ with the induced module and comodule structures.

Exercise 1.1. *Verify that the following two conditions are equivalent:*
(a) μ_M is a comodule morphism.
(b) Δ_M is a module morphism.

If the equivalent conditions (a)-(b) from above hold, then M is called an H*-bimodule*.

The goal of this sheet is to prove the structure theorem for bimodules over a Hopf algebra.

Theorem 1.2 (The structure theorem for bimodules)**.** *Let H be a Hopf algebra and M be an H-bimodule. Consider a subspace $N \subset M$ defined by*

$$N := \{m \in M \mid \Delta_M(m) = 1 \otimes m\}.$$

Then, the multiplication map $\mu_M \colon H \otimes N \to M$ is an isomorphism H-bimodules.

Problem 1.3. *Prove Theorem 1.2.*

The suggested scheme of proof consists of the following four steps:

(a) Define a linear map $\gamma \colon M \to M$ as the following composition

$$M \xrightarrow{\Delta_M} H \otimes M \xrightarrow{S \otimes \mathrm{Id}_M} H \otimes M \xrightarrow{\mu_M} M.$$

Prove that $\gamma(M) \subset N$.

Next, we consider two linear maps $\alpha \colon H \otimes N \to M$ and $\beta \colon M \to H \otimes N$ defined by

$$\alpha(h \otimes n) = \mu_M(h \otimes n) \quad \text{and} \quad \beta(m) = (\mathrm{Id}_H \otimes \gamma)(\Delta_M(m)).$$

(b) Verify $\alpha \circ \beta = \mathrm{Id}_M$.

(c) Verify $\beta \circ \alpha = \mathrm{Id}_{H \otimes N}$.

(d) Verify that α is a comodule morphism.