HOMEWORK 2, EXTRA PROBLEMS 2

1. Coalgebra A°

Let (A, μ, η) be an algebra. We define the set $A^\circ \subset A^*$ as follows:

$$A^\circ := \{\alpha \in A^* | \mathrm{Ker}(\alpha) \text{ contains a cofinite ideal}\},$$

where an ideal $J \subset A$ is called *cofinite* if $\dim(A/J) < \infty$.

Exercise 1.1. *Verify that A° is a vector subspace of A. Show that $A^\circ = A^*$ if $\dim(A) < \infty$.*

Problem 1.2. *Let (A, μ_A, η_A) and (B, μ_B, η_B) be two algebras.*

(a) For any algebra morphism $f\colon A \to B$, show that $f^*(B^\circ) \subset A^\circ$.

(b) Recall the natural embedding $A^* \otimes B^* \subset (A \otimes B)^*$. Prove $A^\circ \otimes B^\circ = (A \otimes B)^\circ$.

(c) Verify $\mu_A^*(A^\circ) \subset A^\circ \otimes A^\circ$.

Recall that if (A, μ, η) is a finite-dimensional algebra, then its dual A^* has a natural coalgebra structure. In general, the dual A^* is not a coalgebra, but the above subspace $A^\circ \subset A^*$ has a natural coalgebra structure, due to the following problem.

Problem 1.3. *Let $\Delta\colon A^\circ \to A^\circ \otimes A^\circ$ be the restriction of μ_A^* (see Problem 1.2(c)), while $\epsilon\colon A^\circ \to \mathbf{k}$ be the restriction of η_A^* (i.e., $\epsilon(\alpha) := \alpha(1)$). Verify that $(A^\circ, \Delta, \epsilon)$ is a coalgebra.*

Actually, A° is the maximal coalgebra of A^* with coproduct induced by μ_A^*:

Problem 1.4. *Let A be an algebra and A^* be endowed with a natural $A{-}A$-bimodule structure via $(a.\alpha.b)(x) = \alpha(bxa)$ for any $a, b, x \in A, \alpha \in A^*$. Fix $f \in A^*$. The following are equivalent:*

(a) $f \in A^\circ$.

(b) $\mu_A^*(f) \subset A^* \otimes A^*$.

(c) $A.f$ is finite-dimensional.

(d) $f.A$ is finite-dimensional.

(e) $A.f.A$ is finite-dimensional.

The following problem gives a more conceptual viewpoint towards $(\ )^\circ$.

Problem 1.5. *Prove that $((\ )^*, (\ )^\circ)$ is a pair of adjoint functors between* Coalg *and* $\mathrm{Alg}^{\mathrm{op}}$.

Hint: Given an algebra A, a coalgebra C, an algebra morphism $f \in \mathrm{Hom}_{\mathrm{Alg}}(A, C^*)$, and a coalgebra morphism $g \in \mathrm{Hom}_{\mathrm{Coalg}}(C, A^\circ)$, define $G(f) \in \mathrm{Hom}_{\mathrm{Coalg}}(C, A^\circ)$ and $F(g) \in \mathrm{Hom}_{\mathrm{Alg}}(A, C^*) = \mathrm{Hom}_{\mathrm{Alg}^{\mathrm{op}}}(C^*, A)$ via

$$G(f)\colon C \longrightarrow (C^*)^\circ \xrightarrow{f^\circ} A^\circ \text{ and } F(g)\colon A \longrightarrow A^{**} \longrightarrow (A^\circ)^* \xrightarrow{g^*} C^*,$$

where we used the fact that the image of the natural map $C \to C^{**}$ is in $(C^*)^\circ$ (prove this!).

Exercise 1.6. *Verify that if H is a bialgebra (respectively, a Hopf algebra), then H° has a natural bialgebra (respectively, Hopf algebra) structure.*

2. Cofree coalgebras

Using the functor $(\ )^\circ$ we can now construct the cofree (cocommutative) coalgebras.

Definition 2.1. *Given a vector space V, a pair (C,π) of a coalgebra and a linear map $\pi\colon C\to V$ is called a* ***cofree coalgebra on*** *V if for any coalgebra D and any linear map $f\colon D\to V$, there exists a unique coalgebra morphism $F\colon D\to C$ such that $\pi\circ F=f$.*

Clearly, a cofree coalgebra on V is unique (up to an isomorphism) if it exists.

Problem 2.2. *Let V be a vector space and $T(V^*)$ be the tensor algebra of the dual vector space. Prove that $(T(V^*)^\circ,\pi)$ is the cofree coalgebra on V^{**}, where π is defined as the composition $T(V^*)^\circ\hookrightarrow T(V^*)^*\xrightarrow{i^*}V^{**}$ and $i\colon V^*\hookrightarrow T(V^*)$ is the canonical inclusion.*

The following is almost tautological:

Exercise 2.3. *Let (C,π) be the cofree coalgebra on the vector space W and V be a vector subspace of W. Define $D\subset C$ as the sum of all subcoalgebras E of C such that $\pi(E)\subset V$. Prove that $(D,\pi_{|D})$ is the cofree coalgebra on V.*

Combining Problem 2.2 with Exercise 2.3 applied to $W=V^{**}$, we deduce:

Theorem 2.4. *For any vector space V, the cofree coalgebra on V exists.*

Let us now *dualize* the notion of a symmetric algebra $S(V)$ on a vector space V.

Definition 2.5. *Given a vector space V, a pair (C,π) of a cocommutative coalgebra and a linear map $\pi\colon C\to V$ is called a* ***cofree cocommutative coalgebra on*** *V if for any cocommutative coalgebra D and any linear map $f\colon D\to V$, there exists a unique coalgebra morphism $F\colon D\to C$ such that $\pi\circ F=f$.*

The existence of cofree cocommutative coalgebras follows from Theorem 2.4, due to:

Exercise 2.6. *Let V be a vector space and $(\tilde{C},\tilde{\pi})$ be the cofree coalgebra on V. Define C as the sum of all cocommutative subcoalgebras E of $\tilde{C}$. Prove that $(C,\pi:=\tilde{\pi}_{|C})$ is the cofree cocommutative coalgebra on V.*

Use [Homework 1, Problem 4] to deduce the following result:

Problem 2.7. *Let (C_1,π_1) and (C_2,π_2) be cofree cocommutative coalgebras on the vector spaces V_1,V_2. Prove that $(C_1\otimes C_2,\pi)$ is the cofree cocommutative coalgebra on $V_1\oplus V_2$, where $\pi\colon C_1\otimes C_2\to V_1\oplus V_2$ is defined via $\pi(c_1\otimes c_2)=(\pi_1(c_1)\epsilon_{C_2}(c_2),\pi_2(c_2)\epsilon_{C_1}(c_1))$.*

If V is a vector space, let $C(V)$ denote the cofree cocommutative coalgebra on V.

Exercise 2.8. *For any linear map $f\colon V\to W$, there is a unique "induced" coalgebra morphism $C(f)\colon C(V)\to C(W)$ such that $\pi\circ F=f\circ\pi$. Moreover $C(f\circ g)=C(f)\circ C(g)$.*

Using this result, we can finally endow $C(V)$ with a Hopf algebra structure.

Problem 2.9. *Consider the linear maps* $\mathrm{diag}\colon V\to V\oplus V,\ m\colon V\oplus V\to V,\ \iota\colon V\to V$ *defined by* $\mathrm{diag}(v)=(v,v),\ m((v,w))=v+w,\ \iota(v)=-v$ *and let $a\colon V\to\{0\},b\colon\{0\}\to V$ be the trivial linear maps. Verify that the induced maps $C(\mathrm{diag}),C(a),C(m),C(b),C(\iota)$ (see Exercise 2.8) endow $C(V)$ with a Hopf algebra structure.*