HOMEWORK 3: $\mathbf{k}_q[x, y]$, $M_q(2)$, and $U(\mathfrak{g})$

1. Let $\mathbb{F}$ be a finite field of order q.

(a) Show that $\binom{n}{k}_q$ equals the number of k-dimensional subspaces in $\mathbb{F}^n$.

(b) Use part (a) to deduce the following equalities (Lemma 1 of Lecture 5):

$$\binom{n}{k}_q = \binom{n}{n-k}_q \text{ and } \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q = \binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k} \binom{n-1}{k-1}_q .$$

2. Consider the linear endomorphisms Z, τ_q, δ_q of algebras $\mathbf{k}[z]$ and $\mathbf{k}[[z]]$ defined by

$$(Zf)(z) = zf(z),\ (\tau_q f)(z) = f(qz),\ (\delta_q f)(z) = (f(qz) - f(z))/(qz - z).$$

(a) Verify the following equalities: $\delta_q \tau_q = q \tau_q \delta_q,\ [\delta_q, Z] = \tau_q,\ \delta_q Z - qZ\delta_q = 1$.

(b) Show that any τ_q-derivation δ of $\mathbf{k}[z]$ (that is $\delta(fg) = \delta(f)g + \tau_q(f)\delta(g)$ for any $f, g \in \mathbf{k}[z]$) is of the form $\delta = P\delta_q$ for some polynomial P. If $\delta\tau_q = q\tau_q\delta$, then P must be a constant.

(c) Assume that q is not a root of unity. Prove that the q-exponential $e_q(z)$ is, up to a multiplicative constant, the only formal series solution of the equation $\delta_q(f) = f$.

3. Assume $yx = qxy$, where q is a root of unity of order d. Prove $(x + y)^d = x^d + y^d$.

4. Let $\Lambda_q[\xi, \eta]$ be the algebra $\mathbf{k}\langle \xi, \eta \rangle / (\xi^2, \eta^2, \xi\eta + q\eta\xi)$. Set

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

where the variables a, b, c, d commute with ξ, η. Assume $q^2 \neq -1$.

(a) Prove that a, b, c, d satisfy the defining six relations of $M_q(2)$ if and only if

$$y'x' = qx'y' \text{ and } (\xi')^2 = (\eta')^2 = \xi'\eta' + q\eta'\xi' = 0.$$

(b) Check that $(a\xi + b\eta)(c\xi + d\eta) = \det_q \cdot \xi\eta$. Let R be an algebra and m, m' be two R-points of $M_q(2)$ such that the entries of m' pairwise commute with the entries of m. Deduce the equality $\mathrm{Det}_q(m'm) = \mathrm{Det}_q(m')\mathrm{Det}_q(m)$ (Proposition 2(ii) of Lecture 4).

(c) Find a $M_q(2)$-comodule-algebra structure on $\Lambda_q[\xi, \eta]$.

5. Assume that q is not a root of unity. Prove that the center of $M_q(2)$ is generated by $\det_q$.

6. (a) For a Lie algebra $\mathfrak{g}$ over a field $\mathbf{k}$ of zero characteristic, prove $\mathrm{Prim}(U(\mathfrak{g})) = \mathfrak{g}$.

(b) Let $\mathbf{k}\langle X \rangle$ denote the free algebra on a set X. Assuming $\mathrm{char}(\mathbf{k}) = 0$, describe $\mathrm{Prim}(\mathbf{k}\langle X \rangle)$.

7. (a) Let $\mathrm{char}(\mathbf{k}) = p > 2$. Prove that the center of $U(\mathfrak{sl}_2)$ is generated by 4 elements.

(b)* Determine all the defining relations between the generators found in part (a).