HOMEWORK 3, EXTRA PROBLEMS 1

1. Bialgebra $M_q(n)$

Fix $q \in \mathbf{k}^*$ and $n \in \mathbb{Z}_{>0}$. First, let us generalize the notion of the quantum plane.

Definition 1.1. *The **quantum polynomial algebra** $\mathbf{k}_q[x_1, \ldots, x_n]$ is the associative algebra generated by $x_1, \ldots, x_n$ with the defining relations $x_j x_i = q x_i x_j$ for any $i < j$.*

Exercise 1.2. *Prove that $\{x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} | k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}\}$ form a $\mathbf{k}$-basis of $\mathbf{k}_q[x_1, \ldots, x_n]$.*

Next, we define the algebra $M_q(n)$ generalizing $M_q(2)$ from the class.

Definition 1.3. *Let $M_q(n)$ be the associative algebra generated by $\{t_{i,j}\}_{i,j=1}^n$ with the following defining relations (for any $i < j$ and $a < b$):*

$$(\star) \quad t_{j,a}t_{i,a} = qt_{i,a}t_{j,a},\ t_{i,b}t_{i,a} = qt_{i,a}t_{i,b},\ t_{i,b}t_{j,a} = t_{j,a}t_{i,b},\ [t_{i,a}, t_{j,b}] = (q^{-1} - q)t_{i,b}t_{j,a}.$$

Let us provide an enlightening alternative viewpoint towards $M_q(n)$.

Problem 1.4. *Given n^2 elements $\{T_{i,j}\}_{i,j=1}^n$ of an algebra R, let us encode them in a single R-valued $n \times n$-matrix $T := \sum_{i,j} T_{i,j} E_{i,j}$. Set $R' := R \otimes \mathbf{k}_q[x_1, \ldots, x_n]$, $R'' := \mathbf{k}_q[x_1, \ldots, x_n] \otimes R$. Finally, define elements $\{x_i'\}_{i=1}^n$ and $\{x_i''\}_{i=1}^n$ of R' and R'', respectively, via*

$$\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = T \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \big(x_1'', \cdots, x_n''\big) = \big(x_1, \cdots, x_n\big) \cdot T.$$

Assuming $q^2 \neq -1$, prove that the following two conditions are equivalent:
(a) The generators $T_{i,j}$ satisfy the relations ($\star$) with $t_{i,j}$ replaced by $T_{i,j}$.
(b) We have $x_j' x_i' = q x_i' x_j'$ and $x_j'' x_i'' = q x_i'' x_j''$ for any $i < j$.

Exercise 1.5. *(a) Prove that there exist algebra morphisms $\Delta\colon M_q(n) \to M_q(n) \otimes M_q(n)$ and $\epsilon\colon M_q(n) \to \mathbf{k}$ uniquely determined by $\Delta(T) = T \otimes T$, $\epsilon(T) = I_n$.*

(b) Verify that $(M_q(n), \Delta, \epsilon)$ is a bialgebra.

Definition 1.6. *The **quantum skew polynomial algebra** $\Lambda_q[\xi_1, \ldots, \xi_n]$ is the associative algebra generated by $\xi_1, \ldots, \xi_n$ with the defining relations $\xi_i^2 = 0$, $\xi_i \xi_j = -q \xi_j \xi_i$ for any $i < j$.*

Our next result generalizes the one for $M_q(2)$ from [Homework 3, Problem 4].

Problem 1.7. *(a) In the context of Problem 1.4, set $\bar{R} := R \otimes \Lambda_q[\xi_1, \ldots, \xi_n]$ and define $\xi_1', \ldots, \xi_n' \in \bar{R}$ via $\begin{pmatrix} \xi_1' \\ \vdots \\ \xi_n' \end{pmatrix} = T \cdot \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$. Prove that the assertions of Problem 1.4 are equivalent to $x_j' x_i' = q x_i' x_j'$, $\xi_i' \xi_j' = -q \xi_j' \xi_i'$, $(\xi_i')^2 = 0$* for any $i < j$.

(b) Find left and right $M_q(n)$-algebra-comodule structures on $\Lambda_q[\xi_1, \ldots, \xi_n]$.

Define the **quantum determinant** of $M_q(n)$ via $\det_q := \sum_{\sigma\in S_n}(-q)^{-l(\sigma)}t_{1,\sigma(1)}\dots t_{n,\sigma(n)}$. Likewise, given elements $\{T_{i,j}\}_{i,j=1}^n$ of an algebra R satisfying ($\star$), we define $\mathrm{Det}_q(T)\in R$.

Problem 1.8. *(a) In the context of Problem 1.7, prove that* $\xi'_1\dots\xi'_n = \mathrm{Det}_q(T)\cdot\xi_1\dots\xi_n$.

(b) Deduce that $\det_q$ *is group-like, that is,* $\Delta(\det_q) = \det_q\otimes\det_q$.

(c) Show that $\det_q = \sum_{\sigma\in S_n}(-q)^{-l(\sigma)}t_{\sigma(1),1}\dots t_{\sigma(n),n}$.

More generally, given $I=\{i_1,i_2,\dots,i_k\}$ and $J=\{j_1,j_2,\dots,j_k\}$ with $1\le i_1<\dots<i_k\le n$, $1\le j_1<\dots<j_k\le n$, define the **quantum minor** $\tilde{t}^I_J := \sum_{\sigma\in S_k}(-q)^{-l(\sigma)}t_{i_1,j_{\sigma(1)}}\dots t_{i_k,j_{\sigma(k)}}$.

Problem 1.9. *Recall the two coactions of* $M_q(n)$ *on* $\Lambda_q[\xi_1,\dots,\xi_n]$ *from Problem 1.7(b).*

(a) Write down the formulas for the images of $\xi_I := \xi_{i_1}\dots\xi_{i_k}$ *under both coactions.*

(b) Define $\tilde{t}_{j,i} := (-q)^{i-j}\cdot\tilde{t}^{\{1,\dots,n\}\backslash\{i\}}_{\{1,\dots,n\}\backslash\{j\}}$. *Prove the equalities* $\sum_j t_{i,j}\tilde{t}_{j,k} = \delta_{i,k}\cdot\det_q = \sum_j \tilde{t}_{i,j}t_{j,k}$.

(c) Deduce that $\det_q$ *is a central element of* $M_q(n)$.

2. Hopf algebras $GL_q(n)$ and $SL_q(n)$

Using the aforementioned group-like central element $\det_q$, we can make the key definition.

Definition 2.1. *Define* $GL_q(n) := M_q(n)[t]/(t\det_q - 1)$ *and* $SL_q(n) := M_q(n)/(\det_q - 1)$.

Due to Problems 1.8 and 1.9, the bialgebra structure on $M_q(n)$ together with the standard bialgebra structure on $\mathbf{k}[t]$ give rise to bialgebra structures on $GL_q(n)$ and $SL_q(n)$.

Problem 2.2. *Prove that* $GL_q(n)$ *and* $SL_q(n)$ *are Hopf algebras with antipode* S *given by* $S(t_{i,j}) = \det_q^{-1}\cdot\tilde{t}_{i,j}$.

Note that the algebras $M_q(n), GL_q(n), SL_q(n)$ are well-defined for $q=1$, and are isomorphic to the classical bialgebras $M(n), GL(n), SL(n)$ (by abuse of our notations, those stay for the algebras of regular functions on the corresponding loci).

3. Bialgebras $M_{p,q}(n)$, $GL_{p,q}(n)$, $SL_{p,q}(n)$

In this section, we discuss 1-parameter generalization of the aforementioned constructions. Fix $p,q\in\mathbf{k}^*$ such that $pq\ne -1$. Consider two quantum polynomial algebras $\mathbf{k}_q[x_1,\dots,x_n]$ and $\mathbf{k}_p[y_1,\dots,y_n]$. Let us be given n^2 elements $\{T_{i,j}\}_{i,j=1}^n$ of an algebra R, encoded into a single matrix $T := \sum_{i,j}T_{i,j}E_{i,j}$. Define $R' := R\otimes\mathbf{k}_q[x_1,\dots,x_n]$ and $R'' := \mathbf{k}_p[y_1,\dots,y_n]\otimes R$. Finally, define elements $\{x'_i\}_{i=1}^n$ and $\{y'_i\}_{i=1}^n$ of R' and R'', respectively, via

$$\begin{pmatrix} x'_1\\ \vdots\\ x'_n\end{pmatrix} = T\cdot\begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix} \quad\text{and}\quad (y'_1,\cdots,y'_n) = (y_1,\cdots,y_n)\cdot T.$$

Problem 3.1. *Assuming* $pq\ne -1$, *provide a list of* $\binom{n^2}{2}$ *relations on* $T_{i,j}$ *which are equivalent to* $x'_jx'_i = qx'_ix'_j$ *and* $y'_jy'_i = py'_iy'_j$ *for any* $i<j$ *(and coincide with ($\star$) for* $p=q$*).*

Definition 3.2. *Let* $M_{p,q}(n)$ *be the associative algebra generated by* $\{t_{i,j}\}_{i,j=1}^n$ *subject to the defining relations obtained in Problem 3.1.*

Problem 3.3. *(a) Prove that $M_{p,q}(n)$ is a bialgebra with the coproduct Δ and the counit ϵ defined by the same formulas as for $M_q(n)$ in Exercise 1.5.*

(b) Verify that $t_{1,1}t_{2,2} - p^{-1}t_{1,2}t_{2,1}$ is a central element of $M_{p,q}(2)$. Use it to define algebras $GL_{p,q}(2)$ and $SL_{p,q}(2)$. Endow both with Hopf algebra structures, i.e., determine antipodes.

(c) Generalize (b) to an arbitrary n.*