HOMEWORK 4: $U_q(\mathfrak{sl}_2)$ AND ROOTS OF UNITY

1. For any $m, n \in \mathbb{Z}_{\geq 0}$, verify the following identity (which we used in Lecture 6):

$$E^m F^n = \sum_{i=0}^{\min(m,n)} \begin{bmatrix} m \\ i \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix} [i]! \cdot F^{n-i} \cdot \prod_{j=1}^{i} [K; i+j-(m+n)] \cdot E^{m-i}.$$

2. Assume that F^N acts trivially on a $U_q(\mathfrak{sl}_2)$-representation V. For $0 \leq r \leq N$, define

$$h_r := \prod_{j=1-r}^{r-1} [K; r-N+j].$$

Prove that $F^{N-r} h_r$ acts trivially on V for any $0 \leq r \leq N$.

In Problems 3–5, q is assumed to be a primitive d-th root of unity ($d > 2$). We also define

$$e := \begin{cases} d & \text{if } d \text{ is odd} \\ d/2 & \text{if } d \text{ is even} \end{cases}.$$

3. Prove that the center of $U_q(\mathfrak{sl}_2)$ is generated by E^e, F^e, K^e, K^{-e}, C.

4. Recall the $U_q(\mathfrak{sl}_2)$-modules $W(\lambda, b)$ introduced in the class (here $\lambda \in \mathbf{k}^\times, b \in \mathbf{k}$).

(a) Prove that $W(\lambda, b)$ is simple if $b \neq 0$ or $\lambda^{2e} \neq 1$.

(b) Prove that $W(\pm q^n, 0)$ $(0 \leq n < e)$ is simple if and only if $n = e - 1$.

(c) Prove that $W(\pm q^n, 0)$ $(0 \leq n < e-1)$ has a unique nontrivial submodule W'.

(d) Verify that the unique irreducible quotient of $W(\pm q^n, 0)$ $(0 \leq n < e)$ is isomorphic to $L(n, \pm)$ from the class.

5. Recall the vector space $V(\lambda, a, b)$ and its endomorphisms $E, F, K^{\pm 1}$ introduced in the class (here $\lambda \in \mathbf{k}^\times, a, b \in \mathbf{k}$).

(a) Verify that these endomorphisms define a $U_q(\mathfrak{sl}_2)$-action on $V(\lambda, a, b)$.

(b) Prove that $V(\lambda, a, b)$ is simple if $b \neq 0$.