HOMEWORK 4, EXTRA PROBLEMS 1

1. Ore extensions

Given an algebra R, let us describe all possible algebra structures on the vector space $R[t]$, which satisfy the following two properties:

(1) the natural inclusion $R \hookrightarrow R[t]$ (given by $x \mapsto x \cdot t^0,\ x \in R$) is an algebra morphism,

(2) $\deg(PQ) = \deg(P) + \deg(Q)$ for any $P, Q \in R[t]$ (we assume $\deg(0) = -\infty$).

Exercise 1.1. *Assume that an algebra structure on $R[t]$ satisfying the above two properties is given. Prove that R has no zero-divisors and there exists a unique injective algebra endomorphism α of R and a unique α-derivation δ of R (that is, δ is a linear endomorphism of R satisfying $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$), such that*

$$(\star) \qquad ta = \alpha(a)t + \delta(a) \text{ for any } a \in R.$$

The inverse is also true.

Problem 1.2. *Let R be an algebra without zero-divisors. Given an injective algebra endomorphism α of R and an α-derivation δ of R, prove that there exists a unique algebra structure on $R[t]$ satisfying the above properties (1), (2), and the formula ($\star$).*

Definition 1.3. *The algebra constructed in Problem 1.2 is denoted $R[t; \alpha, \delta]$ and is called the* ***Ore extension****.*

A few basic properties of these algebras are summarized in the next two problems.

Problem 1.4. *In the setup of Problem 1.2, show that $R[t; \alpha, \delta]$ has no zero-divisors. Verify that as a left R-module, it is free with a basis $\{t^k | k \in \mathbb{Z}_{\geq 0}\}$. Assuming α to be an automorphism, prove that $R[t; \alpha, \delta]$ is also a right free R-module with the same basis $\{t^k | k \in \mathbb{Z}_{\geq 0}\}$.*

Recall that a ring A is called **left Noetherian** if either of the equivalent conditions hold:

◦ any left ideal I of A is finitely generated;

◦ any strictly ascending sequence of left ideals $I_1 \subsetneq I_2 \subsetneq \cdots$ of A is finite.

Problem 1.5. *Let α be an algebra automorphism of R and δ be an α-derivation of R. Prove that if R is left Noetherian, then so is the Ore extension $R[t; \alpha, \delta]$.*

Hint: Given a left ideal I of $R[t; \alpha, \delta]$, consider a collection of subsets $\{I_d\}_{d \in \mathbb{Z}_{\geq 0}}$ of R, consisting of 0 and the leading coefficients of degree d elements of I. Prove that they are left ideals, giving rise to the ascending sequence $I_0 \subseteq \alpha^{-1}(I_1) \subseteq \alpha^{-2}(I_2) \subseteq \cdots$ of left ideals in R.

Recall that a ring A is **Noetherian** if it is left Noetherian and the opposite ring A^{op} is also left Noetherian (equivalently, A is right Noetherian).

Exercise 1.6. *(a) Let R be an algebra without zero-divisors, α be the algebra automorphism of R and δ be the α-derivation of R. Verify that $\delta\alpha^{-1}$ is an α^{-1}-derivation of the opposite algebra R^{op} and prove the following algebra isomorphism: $R[t; \alpha, \delta]^{\mathrm{op}} \cong R^{\mathrm{op}}[t; \alpha^{-1}, -\delta\alpha^{-1}]$.*

(b) Deduce that $R[t; \alpha, \delta]$ is Noetherian if R is.

2. PBW bases

The above theory of Ore extensions can be used to deduce the ring-theoretical properties of the algebras we encountered so far in the class: $\mathbf{k}_q[x,y],\ M_q(2),\ U_q(\mathfrak{sl}_2)$.

Exercise 2.1. *Let α be the automorphism of the polynomial ring $\mathbf{k}[x]$ determined by $\alpha(x) = qx$. Prove that $\mathbf{k}_q[x,y] \cong \mathbf{k}[x][y;\alpha,0]$. Deduce that that the quantum plane $\mathbf{k}_q[x,y]$ is Noetherian, has no zero-divisors, and the set of monomials $\{x^k y^l | k,l \in \mathbb{Z}_{\geq 0}\}$ is its $\mathbf{k}$-basis.*

To prove similar properties for the quantum algebra $M_q(2)$, we present it as an iterated Ore extension of $\mathbf{k}$.

Problem 2.2. *Define the algebras A_1, A_2, A_3 as follows:*

$$A_1 := \mathbf{k}[a],\ A_2 := \mathbf{k}\langle a,b\rangle/(ba - qab),\ A_3 := \mathbf{k}\langle a,b,c\rangle/(ba - qab, ca - qac, cb - bc).$$

(a) Let α_1 be the automorphism of A_1 determined by $\alpha_1(a) = qa$. Prove that $A_2 \cong A_1[b;\alpha_1,0]$. Deduce that $\{a^k b^l | k,l \in \mathbb{Z}_{\geq 0}\}$ is a $\mathbf{k}$-basis of A_2 (this essentially coincides with Exercise 2.1).

(b) Let α_2 be the automorphism of A_2 determined by $\alpha_2(a) = qa,\ \alpha_2(b) = b$. Prove that $A_3 \cong A_2[c;\alpha_2,0]$. Deduce that $\{a^k b^l c^m | k,l,m \in \mathbb{Z}_{\geq 0}\}$ is a $\mathbf{k}$-basis of A_3.

(c) Show that there is a unique algebra automorphism α_3 of A_3 such that $\alpha_3(a) = a,\ \alpha_3(b) = qb,\ \alpha_3(c) = qc$. Verify that the linear endomorphism δ of A_3, defined on the basis by

$$\delta(b^l c^m) = 0 \text{ and } \delta(a^k b^l c^m) = -q^{-1}(1 - q^{2k})a^{k-1}b^{l+1}c^{m+1} \text{ for } k > 0$$

is an α_3-derivation of A_3. Prove that $M_q(2) \cong A_3[d;\alpha_3,\delta]$.

(d) Deduce that $M_q(2)$ is Noetherian, has no zero-divisors, and the set $\{a^k b^l c^m d^n\}_{k,l,m,n \in \mathbb{Z}_{\geq 0}}$ is a $\mathbf{k}$-basis of $M_q(2)$.

In the same spirit, one can present $U_q(\mathfrak{sl}_2)$ as an iterated Ore extension, thus yielding alternative proofs of the results from Lecture 6.

Problem 2.3. *Define the algebras A_1, A_2 as follows:*

$$A_1 := \mathbf{k}[K,K^{-1}],\ A_2 := \mathbf{k}\langle K,K^{-1},F\rangle/(K^{\pm 1}\cdot K^{\mp 1} - 1, KF - q^{-2}FK).$$

(a) Let α_1 be the automorphism of A_1 determined by $\alpha_1(K) = q^2 K$. Prove that $A_2 \cong A_1[F;\alpha_1,0]$. Deduce that $\{F^l K^m | l \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}\}$ is a $\mathbf{k}$-basis of A_2.

(b) Let α_2 be the automorphism of A_2 determined by $\alpha_2(K) = q^{-2}K,\ \alpha_2(F) = F$. Verify that the linear endomorphism δ of A_2, defined on the basis by

$$\delta(K^m) = 0 \text{ and } \delta(F^l K^m) = F^{l-1}\sum_{i=0}^{l-1}\frac{q^{-2i}K - q^{2i}K^{-1}}{q - q^{-1}}K^m \text{ for } l > 0$$

is an α_2-derivation of A_2. Prove that $U_q(\mathfrak{sl}_2) \cong A_2[E;\alpha_2,\delta]$.

(c) Deduce that $U_q(\mathfrak{sl}_2)$ is Noetherian, has no zero-divisors, and the set $\{E^k F^l K^m\}_{k,l \in \mathbb{Z}_{\geq 0}}^{m \in \mathbb{Z}}$ is a $\mathbf{k}$-basis of $U_q(\mathfrak{sl}_2)$.

Recall the algebras $\mathbf{k}_q[x_1,\ldots,x_n]$ and $M_q(n)$ from [Homework 3, Extra Problems 1].

Problem 2.4. *(a) Realize $\mathbf{k}_q[x_1,\ldots,x_n]$ as an iterated Ore extension of $\mathbf{k}$.*

(b) Realize $M_q(n)$ as an iterated Ore extension of $\mathbf{k}$.*