HOMEWORK 5: $U_q(\mathfrak{g})$

1. Recall the algebra $\widetilde{U}_q(\mathfrak{sl}_2)$ from Lecture 5, generated by $\{E, F, K^{\pm 1}, L\}$ with a certain list of the defining relations. In Lecture 5, we established its following properties:

(a) $\widetilde{U}_q(\mathfrak{sl}_2) \cong U_q(\mathfrak{sl}_2)$ for $q \neq \pm 1$,

(b) $\widetilde{U}_{q=1}(\mathfrak{sl}_2)$ is well-defined and is isomorphic to $U(\mathfrak{sl}_2)[K]/(K^2-1)$.

Verify that the resulting algebra isomorphism $U(\mathfrak{sl}_2) \cong \widetilde{U}_{q=1}(\mathfrak{sl}_2)/(K-1)$ is actually a Hopf algebra isomorphism (see [Lecture 9, Remark 1]).

2. Recall the intertwiner $\Theta^f \circ \tau \colon M_2 \otimes M_1 \xrightarrow{\sim} M_1 \otimes M_2$ from [Lecture 8, Theorem 2], depending on the choice of a map $f \colon \tilde{\Lambda} \times \tilde{\Lambda} \to \mathbf{k}^\times$ (with $\tilde{\Lambda} := \{\pm q^n | n \in \mathbb{Z}\}$) satisfying the equalities

$$f(\lambda, \mu) = \lambda \cdot f(\lambda, q^2\mu) = \mu \cdot f(q^2\lambda, \mu) \ \forall \ \lambda, \mu \in \tilde{\Lambda}.$$

(a) Classify all such maps $f \colon \tilde{\Lambda} \times \tilde{\Lambda} \to \mathbf{k}^\times$.

(b) Evaluate the matrix of Θ^f in the case $M_1 = M_2 = L(1, +)$ (choose the natural bases).

(c) Classify all such maps $f \colon \tilde{\Lambda} \times \tilde{\Lambda} \to \mathbf{k}^\times$ (resp. $f \colon \Lambda \times \Lambda \to \mathbf{k}^\times$ with $\Lambda := \{q^n | n \in \mathbb{Z}\}$) which in addition satisfy the equalities of [Lecture 9, Proposition 2]: $f(\lambda, \mu\nu) = f(\lambda, \mu) f(\lambda, \nu)$ and $f(\lambda\mu, \nu) = f(\lambda, \nu) f(\mu, \nu)$ for all $\lambda, \mu, \nu \in \tilde{\Lambda}$ (resp. $\lambda, \mu, \nu \in \Lambda$).

3. For a pair of positive simple roots $\alpha \neq \beta$, recall the following elements $u^{\pm}_{\alpha\beta}$ of $\bar{U}_q(\mathfrak{g})$:

$$u^+_{\alpha\beta} := \sum_{i=0}^{1-a_{\alpha\beta}} (-1)^i \begin{bmatrix} 1-a_{\alpha\beta} \\ i \end{bmatrix}_\alpha E_\alpha^{1-a_{\alpha\beta}-i} E_\beta E_\alpha^i, \ u^-_{\alpha\beta} := \sum_{i=0}^{1-a_{\alpha\beta}} (-1)^i \begin{bmatrix} 1-a_{\alpha\beta} \\ i \end{bmatrix}_\alpha F_\alpha^{1-a_{\alpha\beta}-i} F_\beta F_\alpha^i.$$

Prove [Lecture 10, Lemma 4] claiming the following two formulas in $\bar{U}_q(\mathfrak{g}) \otimes \bar{U}_q(\mathfrak{g})$:

$$\Delta(u^+_{\alpha\beta}) = u^+_{\alpha\beta} \otimes 1 + K_\alpha^{1-a_{\alpha\beta}} K_\beta \otimes u^+_{\alpha\beta}, \ \Delta(u^-_{\alpha\beta}) = u^-_{\alpha\beta} \otimes K_\alpha^{-1+a_{\alpha\beta}} K_\beta^{-1} + 1 \otimes u^-_{\alpha\beta}.$$

4. Verify the following equality for $r \in \mathbb{Z}_{>0}$ (used in the proof of [Lecture 11, Lemma 4]):

$$\sum_{i=0}^{r} (-1)^i \begin{bmatrix} r \\ i \end{bmatrix}_x x^{\pm i(1-r)} = 0.$$

5. Choose $k(\alpha) \in \mathbb{Z}$ and $m(\alpha), n(\alpha) \in \mathbb{Z}_{>0}$ for every positive simple root α, and let I be the left ideal of $U_q(\mathfrak{g})$ generated by $E_\alpha^{m(\alpha)}, F_\alpha^{n(\alpha)}, K_\alpha - q^{k(\alpha)}$. Prove that E_α, F_α act locally nilpotently on the $U_q(\mathfrak{g})$-module $U_q(\mathfrak{g})/I$ (this is [Lecture 12, Lemma 5]).