HOMEWORK 5, EXTRA PROBLEMS 1

The goal of this extra homework is to provide examples with explicit constructions of simple finite dimensional $U_q(\mathfrak{g})$-representations.

1. *Explicit construction of representations $L(r\omega_1)$ in type A_n.*

Consider a polynomial algebra $S = \mathbf{k}[x_1, \ldots, x_{n+1}]$ in $n+1$ indeterminates $\{x_i\}_{i=1}^{n+1}$. For every $1 \leq i \leq n+1$, we define endomorphisms $\mathcal{D}_i, \mathcal{M}_i, X_i$ of S via

$$\begin{aligned}
\mathcal{D}_i &\colon x_1^{m_1} \cdots x_{n+1}^{m_{n+1}} \mapsto [m_i]_q \cdot x_1^{m_1} \cdots x_i^{m_i - 1} \cdots x_{n+1}^{m_{n+1}}, \\
\mathcal{M}_i &\colon x_1^{m_1} \cdots x_{n+1}^{m_{n+1}} \mapsto q^{m_i} \cdot x_1^{m_1} \cdots x_{n+1}^{m_{n+1}}, \\
X_i &\colon x_1^{m_1} \cdots x_{n+1}^{m_{n+1}} \mapsto x_1^{m_1} \cdots x_i^{m_i + 1} \cdots x_{n+1}^{m_{n+1}}.
\end{aligned}$$

(a) Verify $\mathcal{D}_i \circ \mathcal{M}_j = q^{\delta_{i,j}} \mathcal{M}_j \circ \mathcal{D}_i$ and $X_i \circ \mathcal{M}_j = q^{-\delta_{i,j}} \mathcal{M}_j \circ X_i$.

(b) Verify $\mathcal{D}_i \circ X_j = X_j \circ \mathcal{D}_i$ $(i \neq j)$ and $\mathcal{D}_i \circ X_i = q^{-1} X_i \circ \mathcal{D}_i + \mathcal{M}_i$.

For any $1 \leq i \leq n$, define endomorphisms e_i, f_i, k_i of S via

$$e_i = X_i \circ \mathcal{D}_{i+1},\ f_i = X_{i+1} \circ \mathcal{D}_i,\ k_i = \mathcal{M}_i \circ \mathcal{M}_{i+1}^{-1}.$$

(c) Let us label the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of $\mathfrak{sl}_{n+1}$ in the standard way. Show that the assignment $E_{\alpha_i} \mapsto e_i, F_{\alpha_i} \mapsto f_i, K_{\alpha_i} \mapsto k_i$ gives rise to the action of $U_q(\mathfrak{sl}_{n+1})$ on S.

(d) Let $S^r \subset S$ be the subspace of all degree r polynomials. Verify that S^r is a $U_q(\mathfrak{sl}_{n+1})$-submodule of S. Prove $S \simeq L(r\omega_1)$, where ω_1 denotes the first fundamental weight of $\mathfrak{sl}_{n+1}$.

2. *Explicit construction of representations $L(\lambda)$ for minuscule dominant weights λ.*

Recall that a dominant weight $\lambda \in P_+ \backslash \{0\}$ is called *minuscule* if $(\lambda, \alpha) \in \{-d_\alpha, 0, d_\alpha\}$ for every root $\alpha \in \Delta$, where we define $d_\alpha := \frac{(\alpha,\alpha)}{2}$ as always.

(a) Verify that the weights of the simple $U_q(\mathfrak{g})$-module $L(\lambda)$ with the highest weight λ are precisely the conjugates of λ under the Weyl group W, each occurring with multiplicity 1.

Let us now provide an explicit construction of such $L(\lambda)$. Consider a vector space L with a basis $\{x_\mu\}_{\mu \in W(\lambda)}$. We define endomorphisms $e_\alpha, f_\alpha, k_\alpha$ $(\alpha \in \Pi)$ of L as follows:

$$k_\alpha(x_\mu) = q^{(\alpha,\mu)} x_\mu, e_\alpha(x_\mu) = \begin{cases} x_{\mu+\alpha} & \text{if } (\mu, \alpha) = -d_\alpha \\ 0 & \text{otherwise} \end{cases}, f_\alpha(x_\mu) = \begin{cases} x_{\mu-\alpha} & \text{if } (\mu, \alpha) = d_\alpha \\ 0 & \text{otherwise} \end{cases}$$

for any $\mu \in W(\lambda)$.

(b) Show that the assignment $E_\alpha \mapsto e_\alpha, F_\alpha \mapsto f_\alpha, K_\alpha \mapsto k_\alpha$ defines the action of $U_q(\mathfrak{g})$ on L.

(c) Prove that $L \simeq L(\lambda)$. Verify that L is simple even if q is a root of unity.

(d) Derive explicit formulas for the *vector representations* in the classical types A_n, B_n, C_n, D_n (actually, those are just $L(\omega_1)$).

3. *Explicit construction of quantum analogues of the adjoint representations.*

Consider a vector space L with a basis $\{x_\gamma\}_{\gamma\in\Delta}\cup\{h_\beta\}_{\beta\in\Pi}$. Define endomorphism k_α of L via $k_\alpha(h_\beta)=h_\beta, k_\alpha(x_\gamma)=q^{(\alpha,\gamma)}x_\gamma$. Next, we define endomorphisms e_α, f_α of L as follows:

- $e_\alpha(x_\alpha)=0,\ e_\alpha(x_{-\alpha})=h_\alpha,\ e_\alpha(h_\alpha)=[2]_\alpha x_\alpha, e_\alpha(h_\gamma)=\left[-d_\gamma^{-1}(\alpha,\gamma)\right]_\gamma\cdot x_\alpha$ for $\gamma\neq\alpha$,
- $f_\alpha(x_\alpha)=h_\alpha,\ f_\alpha(x_{-\alpha})=0,\ f_\alpha(h_\alpha)=[2]_\alpha x_{-\alpha}, f_\alpha(h_\gamma)=\left[-d_\gamma^{-1}(\alpha,\gamma)\right]_\gamma\cdot x_{-\alpha}$ for $\gamma\neq\alpha$,
- All the remaining roots $\Delta\backslash\{\pm\alpha\}$ split into α-*strings* of the form $\{\gamma,\gamma-\alpha,\ldots,\gamma-m\alpha\}$ with $\gamma+\alpha,\gamma-(m+1)\alpha\notin\Delta$ (note that $m=d_\alpha^{-1}\cdot(\gamma,\alpha)$). For each such α-string, we define

$$e_\alpha(x_{\gamma-i\alpha})=\begin{cases}[m+1-i]_\alpha\cdot x_{\gamma-(i-1)\alpha} & \text{if } 0<i\leq m\\ 0 & \text{if } i=0\end{cases},$$

$$f_\alpha(x_{\gamma-i\alpha})=\begin{cases}[i+1]_\alpha\cdot x_{\gamma-(i+1)\alpha} & \text{if } 0\leq i<m\\ 0 & \text{if } i=m\end{cases}.$$

Show that the assignment $E_\alpha\mapsto e_\alpha, F_\alpha\mapsto f_\alpha, K_\alpha\mapsto k_\alpha$ defines the action of $U_q(\mathfrak{g})$ on L. This is the quantum analogue of the adjoint representation.