HOMEWORK 6: DETAILS FROM LECTURES 13–15

1. Recall the setup of Lecture 13: $K = \mathbb{Q}(v) \supset \mathbb{Q}[v, v^{-1}] = A$, V is either $L(\lambda)$ or $\widetilde{L}(\lambda)$ ($\lambda \in P$–dominant weight), and $V_A := \sum_I AF_I v_\lambda, V_{\mu,A} := \sum_{I:\mathrm{wt}(I)=\lambda-\mu} AF_I v_\lambda, \overline{V}_\mu := V_{\mu,A}/(v-1)$.

Verify that V_μ and $V_{\mu,A}$ are free A-modules. Show that $V_{\mu,A} \otimes_A K \to V_\mu$ are isomorphisms. Deduce the equalities $\dim_K(V_\mu) = \mathrm{rk}_A(V_{A,\mu}) = \dim_{\mathbb{C}}(\overline{V}_\mu)$.

2. Consider an embedding $\mathbb{Q}(v) \hookrightarrow \mathbf{k}$ given by $v \mapsto q \in \mathbf{k}$, where q is transcendental over $\mathbb{Q}$. For a dominant weight $\lambda \in P$, verify that $L(\lambda)_{\mathbb{Q}(v)} \otimes_{\mathbb{Q}(v)} \mathbf{k}$ is a simple $U_{\mathbb{Q}(v)} \otimes_{\mathbb{Q}(v)} \mathbf{k} \cong U_{\mathbf{k}}$-module. Deduce the equality $L(\lambda)_{\mathbb{Q}(v)} \otimes_{\mathbb{Q}(v)} \mathbf{k} \cong L(\lambda)_{\mathbf{k}}$ (thus completing the proof of [Theorem 1 of Lecture 13]).

3. Complete the arguments in our proof of [Theorem 2 of Lecture 13] by proving

(a) $L(\lambda)^* \cong L(-w_0\lambda)$ for dominant $\lambda \in P$,

(b) exhibiting an explicit formula for the isomorphism $V \xrightarrow{\sim} (V^*)^*$ of $U_q(\mathfrak{g})$-modules.

4. (a) Generalize the results on $\widetilde{U}_q(\mathfrak{sl}_2)$ of Lecture 5 by defining an algebra $\widetilde{U}_q(\mathfrak{g})$ generated by $\{E_\alpha, F_\alpha, K_\alpha^{\pm 1}, L_\alpha\}_{\alpha \in \Pi}$ with an explicit list of the defining relations such that

∘ If $q^{d_\alpha} \neq 1$ ($\forall \alpha \in \Pi$), then the assignment $E_\alpha \mapsto E_\alpha, F_\alpha \mapsto F_\alpha, K_\alpha \mapsto K_\alpha$ gives rise to an isomorphism $U_q(\mathfrak{g}) \xrightarrow{\sim} \widetilde{U}_q(\mathfrak{g})$,

∘ For $q = 1$, we have an isomorphism $\widetilde{U}_{q=1}(\mathfrak{g}) \cong U(\mathfrak{g})[\{K_\alpha\}_{\alpha \in \Pi}]/(\{K_\alpha^2 - 1\}_{\alpha \in \Pi})$.

(b) Verify that the latter induces a Hopf algebra isomorphism $\widetilde{U}_{q=1}(\mathfrak{g})/(\{K_\alpha - 1\}_{\alpha \in \Pi}) \simeq U(\mathfrak{g})$.

5. Prove Lemma 5 of Lecture 14: $f_J(E_I K_\mu) = f_J(E_I)$ and $f_J(E_I) = 0$ if $\mathrm{wt}(I) \neq \mathrm{wt}(J)$.

6. (a) Verify the equality $(\sigma(y), \sigma(x)) = (y, x)$ for any $y \in U_q^-, x \in U_q^+$ (here σ is the anti-automorphism of $U_q(\mathfrak{g})$ determined by $E_\alpha \mapsto E_\alpha, F_\alpha \mapsto F_\alpha, K_\alpha \mapsto K_\alpha^{-1}$).

(b) Verify the equality $(S(y), S(x)) = (y, x)$ for any $y \in U_q^{\leq}, x \in U_q^{\geq}$ (here S is the antipode).

7. Prove Lemma 6 of Lecture 15, that is,

$$\mathrm{ad}(E_\alpha)(yK_\lambda x) = yK_\lambda(q^{-(\lambda,\alpha)}E_\alpha x - q^{(\mu-\nu,\alpha)}xE_\alpha) + \frac{(q^{-(\nu-\alpha,\alpha)}r_\alpha(y)K_{\lambda+\alpha} - r'_\alpha(y)K_{\lambda-\alpha})x}{q_\alpha - q_\alpha^{-1}},$$

$$\mathrm{ad}(F_\alpha)(yK_\lambda x) = q^{-(\mu,\alpha)}(F_\alpha y - q^{-(\lambda,\alpha)}yF_\alpha)K_{\lambda+\alpha}x + \frac{y(q^{-(\mu-\alpha,\alpha)}K_\lambda r'_\alpha(x) - q^{-2(\mu-\alpha,\alpha)}K_{\lambda+2\alpha}r_\alpha(x))}{q_\alpha - q_\alpha^{-1}}$$

for any $\alpha \in \Pi, \lambda \in Q, x \in (U_q^+)_\mu, y \in (U_q^-)_{-\nu}$.

8. Verify that Proposition 2 of Lecture 15 is equivalent to the following result: the linear map $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \to \mathbf{k}$ given by $v \otimes v' \mapsto (v, v')$ is a $U_q(\mathfrak{g})$-morphism.