HOMEWORK 7: DETAILS FROM LECTURES 16–17

1. Complete the proof of [Lecture 16, Proposition 1] by:
(a) Working out computational details in the proof of $\langle \mathrm{ad}(E_\alpha)v, v'\rangle = \langle v, \mathrm{ad}(S(E_\alpha))v'\rangle$.
(b) Deduce the equality $\langle \mathrm{ad}(F_\alpha)v, v'\rangle = \langle v, \mathrm{ad}(S(F_\alpha))v'\rangle$ by verifying:

$$\omega \circ S \circ \mathrm{ad}(F_\alpha) = q_\alpha^2 \mathrm{ad}(E_\alpha) \circ \omega \circ S,\ \omega \circ S \circ \mathrm{ad}(S(F_\alpha)) = q_\alpha^2 \mathrm{ad}(-E_\alpha K_\alpha^{-1}) \circ \omega \circ S.$$

2. Verify that the assignment $u \mapsto \mathrm{tr}_{L(\lambda)}(uK_{-2\rho})$ gives rise to a $U_q(\mathfrak{g})$-morphism $U_q(\mathfrak{g}) \to \mathbf{k}$, where the left-hand-side carries the adjoint action, while the right-hand-side is endowed with a trivial module structure (thus completing our argument in [Lecture 16, Lemma 2]).

3. Complete the proof of [Lecture 17, Lemma 2] by verifying the following equality:

$$(1 \otimes F_\alpha)\Theta_\mu + (F_\alpha \otimes K_\alpha^{-1})\Theta_{\mu-\alpha} = \Theta_\mu(1 \otimes F_\alpha) + \Theta_{\mu-\alpha}(F_\alpha \otimes K_\alpha).$$

4. Work out details in the proof of [Lecture 17, Theorem 1].

5. Complete the proof of [Lecture 17, Lemma 3] by verifying the following equality:

$$(1 \otimes \Delta)\Theta_\mu = \sum_{0 \le \nu \le \mu} (\Theta_{\mu-\nu})_{12}(1 \otimes K_\nu \otimes 1)(\Theta_\nu)_{13}.$$

6. Following discussions of Lecture 17, consider the simplest case $\mathfrak{g} = \mathfrak{sl}_2, M = M' = V = L(1,+)$ with $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbf{k}$ determined by $f(1,1) = q^{-1}$.
(a) Verify that $R^2 = (q^{-1} - q)R + 1$, or equivalently, $(qR^{-1})^2 = (q^2-1)(qR^{-1}) + q^2$.
(b) Deduce that the operators $\{R'_i\}_{i=1}^{r-1} \subset \mathrm{End}(V^{\otimes r})$ given by $V'_i = qR_i^{-1}$ satisfy

$$R'_iR'_{i+1}R'_i = R'_{i+1}R'_iR'_{i+1},\ R'_iR'_j = R'_jR'_i\ (j \neq i, i \pm 1),\ (R'_i)^2 = (q^2-1)R'_i + q^2.$$

In other words, $\{R'_i\}_{i=1}^{r-1}$ define a representation of the type A_{r-1} Hecke algebra on $V^{\otimes r}$.

7. Assuming $\mathbf{k}$ contains appropriate roots of q, determine all maps $f\colon P \times P \to \mathbf{k}$ satisfying

$$f(\lambda+\eta, \mu) = q^{-(\eta,\mu)}f(\lambda,\mu),\ f(\lambda, \mu+\eta) = q^{-(\eta,\lambda)}f(\lambda,\mu),$$
$$f(\lambda+\nu, \mu) = f(\lambda,\mu)f(\nu,\mu),\ f(\lambda, \mu+\nu) = f(\lambda,\mu)f(\lambda,\nu)$$

for any $\lambda, \mu, \nu \in P, \eta \in Q$ (under such choices both Theorems 1 and 3 of Lecture 17 do hold).

8. Prove that $k_q[G]$ has a Hopf algebra structure (see suggested formulas in the notes).

9. In the simplest case $G = \mathrm{SL}(2)$, verify:
(a) $k_q[\mathrm{SL}(2)]$ is generated by the matrix coefficients corresponding to $M = L(1,+)$.
(b) Verify that the corresponding four matrix coefficients satisfy the defining relations of $\mathrm{SL}_{q^{-1}}(2)$, thus giving rise to a surjective homomorphism $\phi\colon \mathrm{SL}_{q^{-1}}(2) \to k_q[\mathrm{SL}(2)]$.
(c)* Verify that the homomorphism ϕ of part (b) is actually an isomorphism.