HOMEWORK 8: DETAILS FROM LECTURES 18–20

1. Given any finite-dimensional $\mathfrak{g}$-representation V, consider an endomorphism $\tilde{s}_\alpha \in \mathrm{End}(V)$ defined by $\tilde{s}_\alpha = \exp(e_\alpha)\exp(-f_\alpha)\exp(e_\alpha)$. Verify that it maps V_μ isomorphically onto $V_{s_\alpha \mu}$.

2. Complete the proof of [Lecture 18, Lemma 1] by establishing the following equality:

$$\sum_{0\leq a,c\leq i} (-1)^{b+j} q^{\pm(b-ac-j(i+1))} \begin{bmatrix} i \\ a \end{bmatrix}_q \begin{bmatrix} j+c \\ c \end{bmatrix}_q \begin{bmatrix} j+a \\ i-c \end{bmatrix}_q = 1 \text{ for any } i,j,b \geq 0.$$

Hint: You may want to prove first $\begin{bmatrix} a+b \\ k \end{bmatrix}_q = \sum_{i=0}^{k} q^{ai-b(k-i)} \begin{bmatrix} b \\ i \end{bmatrix}_q \begin{bmatrix} a \\ k-i \end{bmatrix}_q$ for any $a,b,k \geq 0$. Provide both algebraic and combinatoric proofs of the latter equality.

3. Prove Lemma 4 of Lecture 18.

4. Given $\alpha \neq \beta \in \Pi$, a finite-dimensional $U_q(\mathfrak{g})$-representation V, and $v \in V$, verify

$$T_\alpha(E_\beta v) = \left(\mathrm{ad}(E_\alpha^{(r)})E_\beta\right) T_\alpha(v), \text{ where } r := -2(\beta,\alpha)/(\alpha,\alpha).$$

This is Proposition 1 of Lecture 18:

5. Consider an algebra homomorphism ${}^\sigma\mathrm{ad}\colon U_q(\mathfrak{g}) \to \mathrm{End}_{\mathbf{k}}(U_q(\mathfrak{g}))$ given by $x \mapsto \sigma\circ\mathrm{ad}(x)\circ\sigma$, where σ denotes the anti-automorphism of $U_q(\mathfrak{g})$ from Lecture 10.

(a) Prove $T_\alpha({}^\sigma\mathrm{ad}(E_\alpha)u) = \mathrm{ad}(F_\alpha)T_\alpha(u)$ and $T_\alpha({}^\sigma\mathrm{ad}(F_\alpha)u) = \mathrm{ad}(E_\alpha)T_\alpha(u)$ for any $u \in U_q(\mathfrak{g})$.

(b) Deduce that T_α is surjective and obtain explicit formulas for $T_\alpha^{-1}(E_\beta)$ and $T_\alpha^{-1}(F_\beta)$.

6. Given $\alpha \neq \beta \in \Pi$ such that $s_\alpha s_\beta \in W$ is of order 4, prove:

(a) $T_\alpha T_\beta T_\alpha T_\beta = T_\beta T_\alpha T_\beta T_\alpha$ in $\mathrm{Aut}(U_q(\mathfrak{g}))$, cf. [Lecture 19, Theorem 1].

(b) Given $w \in \langle s_\alpha, s_\beta\rangle \subset W$ such that $w\alpha > 0$, verify that $T_w(E_\alpha) \in \langle E_\alpha, E_\beta\rangle \subset U_q^+$ and $T_w(E_\alpha) = E_{w\alpha}$ if $w\alpha \in \Pi$, cf. [Lecture 19, Lemma 1].

(c) The span of the products (*) in Lecture 20, corresponding to the longest element w of $\langle s_\alpha, s_\beta\rangle \subset W$ coincides with the subalgebra $\langle E_\alpha, E_\beta\rangle$ of U_q^+, cf. [Lecture 20, Lemma 1].

7. Given $w \in W$ and $\alpha \neq \beta \in \Pi$ satisfying $w\alpha > 0$ and $w\beta < 0$, prove that there exists a decomposition $w = w' \cdot w''$ such that (1) $l(w) = l(w') + l(w'')$, (2) $w'' \in \langle s_\alpha, s_\beta\rangle$, (3) $w'\alpha > 0$ and $w'\beta > 0$ (this result was used in our proof of [Lecture 19, Proposition 1]).

8. Given $\alpha \in \Pi, x \in U^+[s_\alpha w_0], y \in U^-[s_\alpha w_0]$ and $i,j \in \mathbb{N}$, verify

$$\left(T_\alpha(y)F_\alpha^i, T_\alpha(x)E_\alpha^j\right) = \delta_{i,j}\,(T_\alpha(y), T_\alpha(x))\,(F_\alpha^i, E_\alpha^i).$$

This is Lemma 5 of Lecture 20.