- Coalgebras

Recall that an algebra is given by a vector space A, multiplication $\mu: A\otimes A \longrightarrow A$ and a unit $\eta: k \longrightarrow A$ satisfying the following compatibilities:

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "\mu\otimes Id"] \arrow[d, "Id\otimes\mu"'] & A\otimes A \arrow[d, "\mu"] \\ A\otimes A \arrow[r, "\mu"] & A \end{tikzcd} , \quad \begin{tikzcd} k\otimes A \arrow[r, "\eta\otimes Id"] \arrow[dr, "\simeq"'] & A\otimes A \arrow[d, "\mu"] & A\otimes k \arrow[l, "Id\otimes\eta"'] \arrow[dl, "\simeq"] \\ & A & \end{tikzcd}$$

Finally the commutativity of $\begin{tikzcd} A\otimes A \arrow[rr, "\tau_{A,A}"] \arrow[dr, "\mu"'] & & A\otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$ encodes the condition "A-commutative".

"flip"

Reversing all the arrows, we get a notion of coalgebra.

Def: A coalgebra is a triple (C, Δ, ε), where C is a vector space, $\Delta: C \longrightarrow C\otimes C$, $\varepsilon: C \longrightarrow k$ - linear maps which fit into commutative diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C\otimes C \arrow[d, "Id\otimes\Delta"] \\ C\otimes C \arrow[r, "\Delta\otimes Id"] & C\otimes C\otimes C \end{tikzcd} , \quad \begin{tikzcd} k\otimes C & C\otimes C \arrow[l, "\varepsilon\otimes Id"'] \arrow[r, "Id\otimes\varepsilon"] & C\otimes k \\ & C \arrow[ul, "\simeq"] \arrow[u, "\Delta"] \arrow[ur, "\simeq"'] & \end{tikzcd}$$

If in addition, the diagram $\begin{tikzcd} C\otimes C \arrow[rr, "\tau_{C,C}"] & & C\otimes C \\ & C \arrow[ul, "\Delta"] \arrow[ur] & \end{tikzcd}$ commutes, we say that the coalgebra is cocommutative.

Notation: Δ is called the coproduct/comultiplication
ε is called the counit

Def: Given two coalgebras (C, Δ, ε) and $(C', \Delta', \varepsilon')$, a linear map $f: C \longrightarrow C'$ is a morphism of coalgebras if the following commute

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & C' \arrow[dl, "\varepsilon'"] \\ & k & \end{tikzcd} , \quad \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\ C\otimes C \arrow[r, "f\otimes f"] & C'\otimes C' \end{tikzcd}$$

Ex1: Given a coalgebra (C, Δ, ε), set $\Delta^{op} := \tau_{C,C} \circ \Delta$. Then $(C, \Delta^{op}, \varepsilon)$ is also a coalgebra, called opposite coalgebra, denoted C^{op}.

Ex2: Given two coalgebras (C, Δ, ε), $(C', \Delta', \varepsilon')$, $C\otimes C'$ is naturally equipped with coalgebra structure via $\Delta^{\otimes} = (Id\otimes\tau_{C,C'}\otimes Id)(\Delta\otimes\Delta')$, $\varepsilon^{\otimes} = \varepsilon\otimes\varepsilon'$.

The relation between algebras and coalgebras is the following:

Lemma 1: (a) The dual vector space of a coalgebra is an algebra

(b) The dual vector space of a finite-dimensional algebra is a coalgebra.

▸ (a) Let (C, Δ, ε) be a coalgebra and let $\bar{\lambda}: C^* \otimes C^* \longrightarrow (C \otimes C)^*$ be the natural linear map. Then $(A = C^*, \mu = \Delta^* \circ \bar{\lambda}, \eta = \varepsilon^*)$ is an algebra.

(b) Let (A, μ, η) be an algebra ($\dim A < \infty$). Then $\bar{\lambda}: A^* \otimes A^* \xrightarrow{\simeq} (A \otimes A)^*$

Then $(C = A^*, \Delta = \bar{\lambda}^{-1} \circ \mu^*, \varepsilon = \eta^*)$ is a coalgebra ∎

Ex 3: (a) Given any set X, define $k[X] := \bigoplus_{x \in X} k \cdot x$. It has a coalgebra structure with $\Delta(x) = x \otimes x$, $\varepsilon(x) = 1$ for $x \in X$.

(b) Moreover, given two sets X, Y, there is a natural coalgebra isom.

$$k[X] \otimes k[Y] \simeq k[X \times Y]$$

(c) The dual algebra of $k[X]$ is just the algebra of k-valued functions on X (with the unit given by constant $= 1$ function).

Def: Given a coalgebra (C, Δ, ε), a subspace $I \subset C$ is a coideal if $\Delta(I) \subset I \otimes C + C \otimes I$ and $\varepsilon(I) = 0$.

If I is a coideal of (C, Δ, ε), then C/I is naturally endowed with a coalgebra structure, called the quotient-coalgebra, via the induced maps $\bar{\varepsilon}: C/I \to k$, $\bar{\Delta}: C/I \longrightarrow C/I \otimes C/I$.

Notation (Sweedler's notation): $\Delta(x) = \sum_{(x)} x' \otimes x''$

e.g. (i) coassociativity reads $\sum_{(x)} \left(\sum_{(x')} (x')' \otimes (x')'' \right) \otimes x'' = \sum_{(x)} x' \otimes \left(\sum_{(x'')} (x'')' \otimes (x'')'' \right)$

which is identified for simplicity with $\sum_{(x)} x' \otimes x'' \otimes x'''$.

(ii) counitality reads $\sum_{(x)} \varepsilon(x') x'' = x = \sum_{(x)} x' \varepsilon(x'')$

(iii) cocommutativity reads $\sum_{(x)} x' \otimes x'' = \sum_{(x)} x'' \otimes x'$.

Notation ($\Delta^{(n)}$): For $n = 1$, $\Delta^{(1)} := \Delta: C \to C^{\otimes 2}$, while for $n > 1$, we inductively define $\Delta^{(n)} := (\Delta \otimes \mathrm{Id}_{C^{\otimes(n-1)}}) \circ \Delta^{(n-1)} : C \longrightarrow C^{\otimes(n+1)}$.

• **Bialgebras**

Let us now assume that a vector space H is equipped both with an algebra structure (H, μ, η) and a coalgebra structure (H, Δ, ε). In particular, $H \otimes H$ is also equipped with both an algebra and a coalgebra structure.

Prop 1: The following are equivalent:

(i) The maps μ, η are morphisms of coalgebras

(ii) The maps Δ, ε are morphisms of algebras

▸ The validity of (i) is equivalent to the commutativity of 4 diagrams:

$$\begin{tikzcd} k \arrow[r, "\eta"] \arrow[d, "\mathrm{Id}"'] & H \arrow[d, "\Delta"] \\ k\otimes k \arrow[r, "\eta\otimes\eta"] & H\otimes H \end{tikzcd}, \quad \begin{tikzcd} k \arrow[rr, "\eta"] \arrow[dr, "\mathrm{Id}"'] & & H \arrow[dl, "\varepsilon"] \\ & k & \end{tikzcd}, \quad \begin{tikzcd} H\otimes H \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "\mu"'] & k\otimes k \arrow[d, "\mu"] \\ H \arrow[r, "\varepsilon"] & k \end{tikzcd}, \quad \begin{tikzcd} H\otimes H \arrow[r, "\mu"] \arrow[d, "(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})(\Delta\otimes\Delta)"'] & H \arrow[d, "\Delta"] \\ (H\otimes H)\otimes(H\otimes H) \arrow[r, "\mu\otimes\mu"] & H\otimes H \end{tikzcd}$$

while the validity of (ii) is equivalent to the commutativity of:

$$\begin{tikzcd} H\otimes H \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "\mu"'] & k\otimes k \arrow[d, "\mu"] \\ H \arrow[r, "\varepsilon"] & k \end{tikzcd}, \quad \begin{tikzcd} k \arrow[rr, "\eta"] \arrow[dr, "\mathrm{Id}"'] & & H \arrow[dl, "\varepsilon"] \\ & k & \end{tikzcd}, \quad \begin{tikzcd} k \arrow[r, "\eta"] \arrow[d, "\mathrm{Id}"'] & H \arrow[d, "\Delta"] \\ k\otimes k \arrow[r, "\eta\otimes\eta"] & H\otimes H \end{tikzcd}, \quad \begin{tikzcd} H\otimes H \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "\mu"'] & (H\otimes H)\otimes(H\otimes H) \arrow[d, "(\mu\otimes\mu)(\mathrm{Id}\otimes\tau\otimes\mathrm{Id})"] \\ H \arrow[r, "\Delta"] & H\otimes H \end{tikzcd}$$

And we see that these are essentially the same diagrams! ∎

Def: A bialgebra is a quantuple $(H, \mu, \eta, \Delta, \varepsilon)$, where (H, μ, η) - algebra, (H, Δ, ε) - coalgebra, which satisfy the equivalent conditions of above Prop 1.

A <u>morphism of bialgebras</u> is a linear map, which is both a morphism of underlying algebras and coalgebras.

Note: For a bialgebra we have $\varepsilon(1)=1$, $\Delta(1)=1\otimes 1$, $\varepsilon(xy)=\varepsilon(x)\varepsilon(y)$, $\Delta(xy)=\Delta(x)\Delta(y)$.

Exercise 1: Let $H=(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. Verify that $H^{op} := (H, \mu^{op}, \eta, \Delta, \varepsilon)$, $H^{cop} := (H, \mu, \eta, \Delta^{op}, \varepsilon)$, $H^{op\,cop} := (H, \mu^{op}, \eta, \Delta^{op}, \varepsilon)$ are bialgebras.

Ex 4: Due to Lemma 1, if H is a finite-dimensional bialgebra, then H^* also has a canonical bialgebra structure.

Ex 5: (a) Given a set X with a <u>unital monoid structure</u> ($\mu: X\times X \to X$, $e\in X$ s.t. $\mu(e,x)=\mu(x,e)=x$) the coalgebra $k[X]$ from Ex 3 is naturally equipped with a bialgebra structure.

(b) Moreover, if X is finite, then the induced coproduct and counit on the algebra of k-valued functions are given by

$$\Delta(f)(x\otimes y) = f(xy), \quad \varepsilon(f) = f(e).$$

Given a vector space V, we have the tensor algebra of V, $T(V)=\bigoplus_{n\geq 0} T^n(V)$ with $T^0(V)=k$, $T^n(V)=V^{\otimes n}$ (the product is induced by $T^n(V)\otimes T^m(V)\simeq T^{n+m}(V)$). This algebra satisfies the universal property $\mathrm{Hom}_{Alg}(T(V),A)\simeq \mathrm{Hom}_{Lin}(V,A)$ for any algebra A.

Exercise 2: Given a vector space V, there is a unique bialgebra structure on $T(V)$ such that $\Delta(v)=1\otimes v+v\otimes 1$, $\varepsilon(v)=0$ $\forall v\in V$. This bialgebra is cocommutative. Moreover, for any $v_1,\dots,v_n\in V$ $(n\geq 1)$:

$\varepsilon(v_1\dots v_n)=0$

(†) $\Delta(v_1\dots v_n)=1\otimes v_1\dots v_n+v_1\dots v_n\otimes 1+\sum_{p=1}^{n-1}\sum_{\sigma\text{-shuffle}} v_{\sigma(1)}\dots v_{\sigma(p)}\otimes v_{\sigma(p+1)}\dots v_{\sigma(n)}$,

where the latter sum is over $(p,n-p)$-shuffles, i.e. $\{\sigma\in S_n \mid \begin{smallmatrix}\sigma(1)<\dots<\sigma(p)\\ \sigma(p+1)<\dots<\sigma(n)\end{smallmatrix}\}$.

Hint: To prove coassociativity, use the fact that $\Delta: T(V)\to T(V)\otimes T(V)$ is induced by $\mathrm{diag}: V\to V\oplus V$ $v\mapsto (v,v)$.

Def: An element $x\in C$ of a coalgebra (C,Δ,ε) is primitive if $\Delta(x)=x\otimes 1+1\otimes x$.
$\mathrm{Prim}(C):=$ subspace of all primitive elements of C.

Lemma 2: (i) If x – primitive element of a bialgebra $\Rightarrow \varepsilon(x)=0$.

(ii) If x,y – primitive of bialgebra, then $[x,y]=xy-yx$ – primitive.

▶ (i) $x=(\varepsilon\otimes \mathrm{Id})(\Delta(x))=\varepsilon(x)\cdot 1+\varepsilon(1)x=\varepsilon(x)+x \Rightarrow \varepsilon(x)=0.$

(ii) $$\left.\begin{aligned}\Delta(xy)&=\Delta(x)\Delta(y)=xy\otimes 1+1\otimes xy+x\otimes y+y\otimes x\\ \Delta(yx)&=\Delta(y)\Delta(x)=yx\otimes 1+1\otimes yx+y\otimes x+x\otimes y\end{aligned}\right\}\Rightarrow \Delta([x,y])=[x,y]\otimes 1+1\otimes[x,y]$$

■

Given a bialgebra H and $x_1,\dots,x_n\in \mathrm{Prim}(H)$, consider $V=\bigoplus_{i=1}^n kv_i$. By universality, there is a unique algebra morphism $f:T(V)\to H$ s.t. $v_i\mapsto x_i$.

Lemma 3: The map $f:T(V)\to H$ is a morphism of bialgebras.
In particular, $\Delta(x_1\dots x_n)$ is given by (†) as above.

▶ Need to check $\varepsilon(f(w))=\varepsilon(w)$, $(f\otimes f)(\Delta(w))=\Delta(f(w))$ $\forall w\in T(V)$.
It suffices to verify these for $w\in V$, whence it follows from Lemma 2(i) ■

• Hopf algebras

Given an algebra (A, μ, η) and a coalgebra (C, Δ, ε), we define a bilinear map "convolution" on $\mathrm{Hom}_{Lin}(C, A)$ as follows:

$$C \xrightarrow{\Delta} C\otimes C \xrightarrow{f\otimes g} A\otimes A \xrightarrow{\mu} A \quad \text{equiv.} \quad (f\star g)(x)=\sum_{(x)} f(x')g(x'')$$

Prop 2: (a) $(\mathrm{Hom}(C,A), \star, \eta\circ\varepsilon)$ is an algebra.

(b) The natural map $A\otimes C^* \xrightarrow{\bar{\lambda}} \mathrm{Hom}(C,A)$ is a morphism of algebras, where C^* is viewed as an algebra via Lemma 1(a).

▶ (a) Due to associativity of μ and coassociativity of Δ, we get:

$$((f\star g)\star h)(x) = \sum_{(x)} f(x')g(x'')h(x''') = (f\star(g\star h))(x)$$

We also have:

$$((\eta\varepsilon)\star f)(x) = \sum_{(x)} \varepsilon(x')f(x'') = f\Big(\sum_{(x)}\varepsilon(x')x''\Big) = f(x)$$

$$(f\star(\eta\varepsilon))(x) = \sum_{(x)} f(x')\varepsilon(x'') = f\Big(\sum_{(x)} x'\varepsilon(x'')\Big) = f(x).$$

(b) To check that $\bar{\lambda}$ is compatible with products, we argue:

$$(\bar{\lambda}(a\star\alpha)\star\bar{\lambda}(b\star\beta))(x) = \sum_{(x)}\alpha(x')\beta(x'')ab = (\alpha\beta)(x)\,ab = (\bar{\lambda}(ab\otimes\alpha\beta))(x)$$

for any $a,b\in A$, $\alpha,\beta\in C^*$, $x\in C$. To see the compatibility of units:

$(\bar{\lambda}(1\otimes\varepsilon))(x) = \varepsilon(x)1 = \eta\varepsilon(x)$ ∎

– LECTURE #2 — (01/22/2018)

Def: Let $(H,\mu,\eta,\Delta,\varepsilon)$ be a bialgebra and $\star$ be the convolution on $\mathrm{End}(H)$. An element $S\in\mathrm{End}(H)$ is called an *antipode* if $S\star \mathrm{Id}_H = \mathrm{Id}_H\star S = \eta\circ\varepsilon$.

Def: A Hopf algebra is a bialgebra with an antipode.
A morphism of Hopf algebras is a morphism of bialgebras commuting with the antipodes

Note: If an antipode exists, then it is unique, due to:

$$S = S\star(\eta\varepsilon) = S\star(\mathrm{Id}_H\star S') = (S\star \mathrm{Id}_H)\star S' = (\eta\varepsilon)\star S' = S' \quad \text{for two antipodes } S, S'$$

Notation: We will denote Hopf algebras by tuples $(H,\mu,\eta,\Delta,\varepsilon,S)$.

Note that in Sweedler's notation, conditions on antipode S read as follows: $\boxed{\sum_{(x)} S(x')x'' = \varepsilon(x)1 = \sum_{(x)} x'S(x'')}$ $(\diamond)$