• Hopf algebras

Given an algebra (A,μ,η) and a coalgebra (C,Δ,ε), we define a bilinear map "convolution" on $\mathrm{Hom}_{Lin}(C,A)$ as follows:

$$C \xrightarrow{\Delta} C\otimes C \xrightarrow{f\otimes g} A\otimes A \xrightarrow{\mu} A \quad \text{equiv.} \quad (f*g)(x)=\sum_{(x)} f(x')g(x'')$$

Prop 2: (a) $(\mathrm{Hom}(C,A), *, \eta\circ\varepsilon)$ is an algebra.

(b) The natural map $A\otimes C^* \xrightarrow{\bar{\lambda}} \mathrm{Hom}(C,A)$ is a morphism of algebras, where C^* is viewed as an algebra via Lemma 1(a).

▶ (a) Due to associativity of μ and coassociativity of Δ, we get:

$$((f*g)*h)(x)=\sum_{(x)} f(x')g(x'')h(x''')=(f*(g*h))(x)$$

We also have:

$$((\eta\varepsilon)*f)(x)=\sum_{(x)}\varepsilon(x')f(x'')=f\Big(\sum_{(x)}\varepsilon(x')x''\Big)=f(x)$$

$$(f*(\eta\varepsilon))(x)=\sum_{(x)} f(x')\,\varepsilon(x'')=f\Big(\sum_{(x)}x'\varepsilon(x'')\Big)=f(x).$$

(b) To check that $\bar{\lambda}$ is compatible with products, we argue:

$$(\bar{\lambda}(a*\alpha)*\bar{\lambda}(b*\beta))(x)=\sum_{(x)}\alpha(x')\beta(x'')ab=(\alpha\beta)(x)\,ab=(\bar{\lambda}(ab\otimes\alpha\beta))(x)$$

for any $a,b\in A$, $\alpha,\beta\in C^*$, $x\in C$. To see the compatibility of units:

$(\bar{\lambda}(1\otimes\varepsilon))(x)=\varepsilon(x)1=\eta\varepsilon(x)$ ■ 5

— LECTURE #2 — (01/22/2018)

Def: Let $(H,\mu,\eta,\Delta,\varepsilon)$ be a bialgebra and $*$ be the convolution on $\mathrm{End}(H)$. An element $S\in\mathrm{End}(H)$ is called an antipode if $S*\mathrm{Id}_H=\mathrm{Id}_H*S=\eta\circ\varepsilon$.

Def: A Hopf algebra is a bialgebra with an antipode.
A morphism of Hopf algebras is a morphism of bialgebras commuting with the antipodes

Note: If an antipode exists, then it is unique, due to:

$$S=S*(\eta\varepsilon)=S*(\mathrm{Id}_H*S')=(S*\mathrm{Id}_H)*S'=(\eta\varepsilon)*S'=S' \quad \text{for two antipodes } S,S'$$

Notation: We will denote Hopf algebras by tuples $(H,\mu,\eta,\Delta,\varepsilon,S)$.

Note that in Sweedler's notation, conditions on antipode S read as follows:

$$\boxed{\sum_{(x)} S(x')x''=\varepsilon(x)1=\sum_{(x)}x'S(x'').} \quad (\diamond)$$

Lemma 1: Let H be a finite-dimensional algebra with antipode S. Then the bialgebra H^* is a Hopf algebra with antipode S^*.

► For any $x \in H$, $\alpha \in H^*$, we have:

$$\Big(\sum_{(\alpha)} \alpha' S^*(\alpha'')\Big)(x) = \sum_{(\alpha)(x)} \alpha'(x') S^*(\alpha'')(x'') = \sum_{(\alpha)(x)} \alpha'(x')\alpha''(Sx'') =$$

$$= \alpha\Big(\sum_{(x)} x' S(x'')\Big) = \alpha(\eta\varepsilon(x)) = \varepsilon^*\eta^*(\alpha)(x)$$

Completely analogously: $\Big(\sum_{\alpha} S^*(\alpha')\alpha''\Big)(x) = \varepsilon^*\eta^*(\alpha)(x)$ ■

Ex 1: If X is a monoid, then $k[X]$ is a bialgebra, due to Ex 5(a). Then, $k[X]$ has an antipode iff X is a group (as $(\diamond)$ reads $xS(x) = S(x)x = \varepsilon(x) = 1$).

The following result summarizes a few properties of antipode.

Prop 1: Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra.

(a) S is a bialgebra morphism from H to $H^{op\,cop}$, i.e.

$$S(xy) = S(y)S(x),\ S(1) = 1,\ (S\otimes S)\Delta = \Delta^{op}S,\ \varepsilon\circ S = \varepsilon$$

(b) TFAE

(i) $S^2 = Id_H$

(ii) $\sum_{(x)} S(x'')x' = \varepsilon(x)1 \quad \forall x \in X.$

(iii) $\sum_{(x)} x'' S(x') = \varepsilon(x)1 \quad \forall x \in X.$

(c) If H is commutative or cocommutative, then $S^2 = Id_H$.

► (b) The implication (i) $\Rightarrow$ (ii) follows from

$$\sum_{(x)} S(x'')x' \underset{(i)}{=} S^2\Big(\sum_{(x)} S(x'')x'\Big) \underset{(a)}{=} S\Big(\sum_{(x)} S(x')S^2(x'')\Big) \underset{(i)}{=} S\Big(\sum_{(x)} S(x')x''\Big) = S(\varepsilon(x)1) = \varepsilon(x)1$$

To prove (ii) $\Rightarrow$ (i) it suffices to prove $S * S^2 = \mu\varepsilon$, due to uniqueness of inverse.

$$(S * S^2)(x) = \sum_{(x)} S(x')S^2(x'') \underset{(a)}{=} S\Big(\sum_{(x)} S(x'')x'\Big) \underset{(ii)}{=} S(\varepsilon(x)1) = \varepsilon(x)S(1) = \varepsilon(x)1$$

The equivalence (i) $\Leftrightarrow$ (iii) is proved analogously.

(c) If H is commutative, then $\eta\varepsilon(x) = \sum_{(x)} x' S(x'') \underset{comm}{=} \sum_{(x)} S(x'')x' \overset{(b)}{\Longrightarrow} S^2 = Id_H$

If H is cocommutative, then $\eta\varepsilon(x) = \sum_{(x)} S(x')x'' \underset{cocomm}{=\!=} \sum_{(x)} S(x'')x' \overset{(b)}{\Longrightarrow} S^2 = Id_H.$

It remains to prove part (a), which is a bit more technical.

(Continuation of proof of Prop 1)

(a) First, plugging $x=1$ into $(\mathrm{Id}_H \star S)(x) = \eta\varepsilon(x)$, we immediately get $S(1)=1$.

Second, $\varepsilon(S(x)) = \varepsilon(S(\sum_{(x)} \varepsilon(x')x'')) = \varepsilon(\sum_{(x)} \varepsilon(x') S(x'')) = \varepsilon(\eta\varepsilon(x)) = \varepsilon(x)$

Next, we prove $S(xy) = S(y)S(x)$. Define $\nu, \rho \in \mathrm{Hom}(H\otimes H, H)$ via
$$\nu(x\otimes y) = S(y)S(x), \quad \rho(x\otimes y) = S(xy).$$
As $\eta\varepsilon$ is the identity (see Prop 2(a)), it suffices to prove $\rho \star \mu = \mu \star \tau = \eta\varepsilon$.

- $(\rho\star\mu)(x\otimes y) = \sum_{(x\otimes y)} \rho((x\otimes y)')\,\mu((x\otimes y)'') = \sum_{(x)(y)} \rho(x'\otimes y')\,\mu(x''\otimes y'') = \sum_{(x)(y)} S(x'y')x''y''$
$$\xlongequal{\Delta(xy)=\Delta(x)\Delta(y)} \sum_{(x)(y)} S((xy)')\,(xy)'' = \eta\varepsilon(xy)$$

- $(\mu\star\nu)(x\otimes y) = \sum_{(x\otimes y)} \mu((x\otimes y)')\,\nu((x\otimes y)'') = \sum_{(x)(y)} \mu(x'\otimes y')\,\nu(x''\otimes y'') =$
$$= \sum_{(x)(y)} x'y'S(y'')S(x'') = \sum_{(x)} x'\eta\varepsilon(y)S(x'') = \eta\varepsilon(y)\cdot\eta\varepsilon(x) = \eta\varepsilon(xy)$$

This completes the proof of $S(xy) = S(y)S(x)$.

Finally, we need to prove $(S\otimes S)\Delta = \Delta^{op}S$, or equivalently:
$\Delta\circ S = (S\otimes S)\circ\Delta^{op}$. Denote $\rho := \Delta\circ S$, $\nu := (S\otimes S)\circ\Delta^{op} \in \mathrm{Hom}(H, H\otimes H)$.

As above, it suffices to prove $\rho\star\Delta = \Delta\star\nu = (\eta\otimes\eta)\varepsilon$

- $(\rho\star\Delta)(x) = \sum_{(x)} \Delta(S(x'))\Delta(x'') = \Delta(\sum_{(x)} S(x')x'') = \Delta(\eta\varepsilon(x)) = ((\eta\otimes\eta)\varepsilon)(x)$

- $(\Delta\star\nu)(x) = \sum_{(x)} \Delta(x')((S\otimes S)(\Delta^{op}(x''))) = \sum_{(x)} (x'\otimes x'')(S(x'''')\otimes S(x'''))$
$$= \sum_{(x)} x'S(x'''')\otimes x''S(x''') = \sum_{(x)} x'S(x''')\otimes\varepsilon(x'')1 =$$
$$= \sum_{(x)} x'\varepsilon(x'')S(x''')\otimes 1 = \sum_{(x)} x'S(x'')\otimes 1 = \varepsilon(x)1\otimes 1 = ((\eta\otimes\eta)\varepsilon)(x)$$

∎

Corollary: Let $H = (H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra.

(a) Then $H^{op\,cop} = (H, \mu^{op}, \eta, \Delta^{op}, \varepsilon, S)$ is also a Hopf algebra and $S: H \to H^{op\,cop}$ is a morphism of Hopf algebras.

(b) If S is an isomorphism, then $H^{op} = (H, \mu^{op}, \eta, \Delta, \varepsilon, S^{-1})$, $H^{cop} = (H, \mu, \eta, \Delta^{op}, \varepsilon, S^{-1})$ are isomorphic Hopf algebras, the isomorphism given by S.

As it is not always easy to check the conditions $\sum_{(x)} S(x')x'' = \varepsilon(x)1 = \sum_{(x)} x'S(x'')$, the following result will be helpful:

Lemma 2: Let H be a bialgebra and $S: H \to H^{op}$ be an algebra morphism. Assume H is generated as an algebra by a subset X and

$$(\Diamond) \quad \sum_{(x)} x' S(x'') = \varepsilon(x)1 = \sum_{(x)} S(x')x'' \quad \forall x \in X$$

Then, S is an antipode for H.

► It suffices to prove that if $(\Diamond)$ holds for $x, y \Rightarrow (\Diamond)$ also holds for xy.

$$\sum_{(xy)} (xy)' S((xy)'') = \sum_{(x)(y)} x'y'S(x''y'') = \sum_{(x)(y)} x'y'S(y'')S(x'') = \sum_{(x)} x'\varepsilon(y)S(x'')$$
$$= \varepsilon(x)\varepsilon(y)1 = \varepsilon(xy)1$$

$$\sum_{(xy)} S((xy)')(xy)'' = \sum_{(x)(y)} S(x'y')x''y'' = \sum_{(x)(y)} S(y')S(x')x''y'' = \sum_{(y)} S(y')\varepsilon(x)y''$$
$$= \varepsilon(x)\varepsilon(y)1 = \varepsilon(xy)1$$
■

Exercise 1: Recall the bialgebra structure on $T(V)$ from [Lecture 1, Exercise 2]. Prove that $S(1)=1$, $S(v_1 \dots v_k) = (-1)^k v_k \dots v_2 v_1$ $\forall v_1, \dots, v_k \in V$ determine an antipode of $T(V)$.

Let us now consider a symmetric algebra $S(V) := T(V)/I(V)$, where $I(V)$ is the 2-sided ideal generated by all elements $\{xy - yx \mid x, y \in V\}$.

Ex 2: The above Hopf algebra structure on $T(V)$ induces a Hopf algebra structure on its quotient $S(V)$.

► It suffices to prove that $I := \mathrm{Ker}\,(T(V) \twoheadrightarrow S(V))$ is a coideal of $T(V)$. Note that I is spanned by the elements of the form $\{x \cdot [v_1, v_2] \cdot y \mid x, y \in T(V),\ v_1, v_2 \in V\}$.
But $\Delta(x[v_1,v_2]y) = \sum_{(x)(y)} (x'[v_1,v_2]y' \otimes x''y'' + x'y' \otimes x''[v_1,v_2]y'') \in I \otimes T(V) + T(V) \otimes I$,

$\varepsilon(x[v_1,v_2]y) = \varepsilon(x)\,\varepsilon([v_1,v_2])\,\varepsilon(y) = 0.$
■

Def: An element $x \neq 0$ of a coalgebra (H, Δ, ε) is grouplike if $\Delta(x) = x \otimes x$.
$G(H) :=$ set of grouplike elements.

Lemma 3: Let H be a bialgebra. Then $G(H)$ is a monoid w.r.t. multiplication of H. Furthermore, if H is a Hopf alg, $x \in G(H) \Rightarrow S(x) \in G(H)$ and $xS(x) = 1 = S(x)x$

► As $\Delta^{op} \circ S = (S \otimes S)(\Delta(x)) \Rightarrow \Delta(S(x)) = S(x) \otimes S(x)$.
By def-n of S: $xS(x) = S(x)x = \varepsilon(x) \cdot 1$. But $x = \varepsilon(x)x = x\varepsilon(x) \Rightarrow \varepsilon(x) \in \{0,1\} \overset{x \neq 0}{\Rightarrow} \varepsilon(x) = 1$ ■

Hopf algebra modules

Since a Hopf algebra is an algebra, we can speak about its modules.

- If U, V are modules over a bialgebra A, then $U\otimes V$ is also an A-module. The latter is given via $\Delta: A \to A\otimes A$ and $A\otimes A \curvearrowright U\otimes V$. In Sweedler's notation

$$\boxed{a(u\otimes v) = \sum_{(a)} a'u\otimes a''v}$$

- Any vector space V is equipped with a trivial A-module structure via counit

$$\boxed{a(v) = \varepsilon(a)\cdot v}$$

Lemma 4: If A is a bialgebra, U, V, W – A-modules, k – trivial A-module, then

$$(U\otimes V)\otimes W \simeq U\otimes(V\otimes W) \quad , \quad k\otimes V \simeq V \simeq V\otimes k \quad \text{as } A\text{-modules}$$

Moreover, if A is cocommutative, then $\tau_{V,W}: V\otimes W \xrightarrow{\sim} W\otimes V$

► Follows immediately from the coassociativity, counitality, cocommutativity. ■

- If A is a Hopf algebra, then given A-modules V, V', the space $\mathrm{Hom}_{\mathrm{Lin}}(V, V')$ is naturally equipped with the A-module structure as follows:
  - First, we endow $\mathrm{Hom}(V, V')$ with an $A\otimes A^{op}$-module structure via

$$((a\otimes a')f)(v) = af(a'v)$$

Indeed:

$$((a\otimes a')(b\otimes b')f)(v) = ((ab\otimes b'a')f)(v) = abf(b'a'v) = a((b\otimes b')f)(a'v) = (a\otimes a'((b\otimes b')f))(v)$$

  - Second, we have an algebra homomorphism $(\mathrm{Id}\otimes S)\circ\Delta: A \to A\otimes A^{op}$.

In Sweedler's notation, the resulting A-action on $\mathrm{Hom}(V, V')$ is given via

$$\boxed{(af)(v) = \sum_{(a)} a'f(S(a'')v)}$$

- In the particular case $V' = k$ with a trivial A-module structure, we obtain an A-action on the dual vector space V^*:

$$(af)(v) = \sum_{(a)} \varepsilon(a')f(S(a'')v) = f\Big(\sum_{(a)} \varepsilon(a')S(a'')v\Big) = f\Big(S\Big(\sum_{(a)} \varepsilon(a')a''\Big)v\Big) = f(S(a)v)$$

So: $A \curvearrowright V^*$ via $\boxed{(af)(v) = f(S(a)v)}$