Lemma 1: Let $(A,\mu,\eta,\Delta,\varepsilon,S)$ be a Hopf algebra and U, U', V, V' be A-modules such that U or U' and V or V' are finite-dimensional vector spaces. Then the linear map

$$\lambda: \mathrm{Hom}(U,U')\otimes \mathrm{Hom}(V,V') \longrightarrow \mathrm{Hom}(V\otimes U, U'\otimes V')$$
$$\lambda(f\otimes g)(v\otimes u) := f(u)\otimes g(v)$$

is A-linear if in addition the flip $\tau_{U^*,V'}: U^*\otimes V' \longrightarrow V'\otimes U^*$ is A-linear. In particular, the following are A-linear:

$$\lambda: U^*\otimes V^* \longrightarrow (V\otimes U)^* \quad \text{and} \quad \lambda_{U,V}: V\otimes U^* \longrightarrow \mathrm{Hom}(U,V)$$
$$v\otimes\alpha \longmapsto (u\mapsto \alpha(u)v)$$

► Fix $f: U\to U'$, $g: V\to V'$, $u\in U$, $v\in V$, $a\in A$. Then:

- $\lambda(a(f\otimes g))(v\otimes u) = \sum_{(a)} \lambda(a'f\otimes a''g)(v\otimes u) = \sum_{(a)} (a'f)(u)\otimes(a''g)(v)$
$$= \sum_{(a)} (a')'f(S((a')'')u)\otimes(a'')'g(S(a'')''v)$$
$$= \sum_{(a)} a'f(S(a'')u)\otimes a'''g(S(a'''')v) \qquad (1)$$

- $(a\lambda(f\otimes g))(v\otimes u) = \sum_{(a)} a'\lambda(f\otimes g)(S(a'')(v\otimes u)) = \sum_{(a)} a'\lambda(f\otimes g)(S(a'')'v\otimes S(a'')''u)$
$$\overset{\text{due to } \Delta S(x)=(S\otimes S)\circ\Delta^{op}}{=} \sum_{(a)} a'\lambda(f\otimes g)(S(a''')v\otimes S(a'')u)$$
$$= \sum_{(a)} a'(f(S(a'')u)\otimes g(S(a''')v)) = \sum_{(a)} (a')'f(S(a'')u)\otimes(a')''g(S(a''')v)$$
$$= \sum_{(a)} a'f(S(a''')u)\otimes a''g(S(a'''')v) \qquad (2)$$

Note that, in general, these formulas prove $\lambda(a(f\otimes g)) \neq a\lambda(f\otimes g)$.

Case 1: $V' = k$ – trivial A-module. Then we replace a''' by $\varepsilon(a''')$ in (1), a'' by $\varepsilon(a'')$ in (2).

Using the equality $\sum_{(x)} \varepsilon(x')x'' = \sum_{(x)} x'\varepsilon(x'') = x$, we see $(1)=(2)$.

Therefore, $\lambda: \mathrm{Hom}(U,U')\otimes V^* \longrightarrow \mathrm{Hom}(V\otimes U, U')$ is A-linear proving the last two claims of Lemma.

Case 2 (general): Follows from the previous case and the following commut. diagram

$$\begin{tikzcd}
\mathrm{Hom}(U,U')\otimes\mathrm{Hom}(V,V') \arrow[d, "\lambda"] & U'\otimes U^*\otimes V'\otimes V^* \arrow[l, "\lambda_{U,U'}\otimes\lambda_{V,V'}"', "\cong \text{ due to fin. dim. conditions}"] \arrow[dr, "Id\otimes\tau_{U^*,V'}\otimes Id"] & \\
 & & U'\otimes V'\otimes U^*\otimes V^* \arrow[dl] \\
\mathrm{Hom}(V\otimes U, U'\otimes V') & U'\otimes V'\otimes(V\otimes U)^* \arrow[l, "\lambda_{V\otimes U, U'\otimes V'}"] &
\end{tikzcd}$$

■

Rmks: (a) If at least one of the pairs (U,U'), (V,V'), (U,V) consists of fin. dim. vector spaces, then $\lambda: \mathrm{Hom}(U,U')\otimes \mathrm{Hom}(V,V') \to \mathrm{Hom}(V\otimes U, U'\otimes V')$ is an isomorphism.

(b) The map $\lambda: U^*\otimes V^* \to (V\otimes U)^*$ is isom. if U or V are fin. dim.

(c) The map $\lambda_{U,V}: V\otimes U^* \to \mathrm{Hom}(U,V)$ is isom. if U or V are fin. dim.

Corollary: If A is cocommutative, then λ from previous Lemma is A-linear.

Lemma 2: Let V be an A-module.

(a) The evaluation map $ev_V: V^*\otimes V \to k$, $\alpha\otimes v \mapsto \alpha(v)$ is A-linear.

(b) If $\dim V<\infty$, then the coevaluation map $\delta_V: k \to V\otimes V^*$, $1 \mapsto \sum_i v_i\otimes v^i$ is A-linear and the composition $\mathrm{Hom}(V,W)\otimes \mathrm{Hom}(U,V) \xrightarrow{\circ} \mathrm{Hom}(U,W)$ is A-linear.

▶ (a) Fix $a\in A$, $v\in V$, $\alpha\in V^*$. Then

$$ev_V(a(\alpha\otimes v)) = \sum_{(a)} ev_V(a'\alpha\otimes a''v) = \sum_{(a)} a'\alpha(a''v) = \alpha\Big(\sum_{(a)} S(a')a''v\Big) =$$
$$= \alpha(\varepsilon(a)\cdot v) = \varepsilon(a)\cdot\alpha(v) = a\cdot ev_V(\alpha\otimes v)$$

which completes the proof.

(b) Note that coevaluation δ_V can be presented as a composition

$$k \xrightarrow[\text{given by unit}]{\eta} \mathrm{End}(V) \xrightarrow{\lambda_{V,V}^{-1}} V\otimes V^* \qquad (\text{here } \eta: 1\mapsto \mathrm{Id}_V)$$

As we already proved $\lambda_{V,V}$ - A-linear. Hence, suffices to prove η is A-linear. This follows from:

$$(a\,\eta(1))(v) = (a\,\mathrm{Id}_V)(v) = \sum_{(a)} a'\,\mathrm{Id}_V(S(a'')v) = \sum_{(a)} a'S(a'')v = \varepsilon(a)v = \eta(\varepsilon(a))(v)$$

Finally, to prove the A-linearity of the composition, we factor it as:

$$\begin{array}{ccc} \mathrm{Hom}(V,W)\otimes\mathrm{Hom}(U,V) & \xrightarrow{\quad\circ\quad} & \mathrm{Hom}(U,W) \\ {\scriptstyle \lambda_{V,W}\otimes\lambda_{U,V}}\uparrow{\scriptstyle \wr\!\wr\ \dim V<\infty} & & \uparrow{\scriptstyle \lambda_{U,W}} \\ W\otimes V^*\otimes V\otimes U^* & \xrightarrow[\mathrm{Id}\otimes ev_V\otimes \mathrm{Id}]{} & W\otimes U^* \end{array}$$

■

Comodules

The notion of comodules over coalgebras is dual to modules over algebras. The latter is a pair $\{(M, \mu_M) \mid M\text{-vector space}, \mu_M : A\otimes M \xrightarrow[\text{linear}]{} M\}$ s.t.

$$\begin{tikzcd} A\otimes A\otimes M \arrow[r, "\mu\otimes Id"] \arrow[d, "Id\otimes\mu_M"'] & A\otimes M \arrow[d, "\mu_M"] \\ A\otimes M \arrow[r, "\mu_M"] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} k\otimes M \arrow[r, "\eta\otimes Id"] \arrow[dr, "\simeq"'] & A\otimes M \arrow[d, "\mu_M"] \\ & M \end{tikzcd} \quad \text{commute}$$

Reversing arrows, we get the following notion:

Def: Let (C, Δ, ε) be a coalgebra. A left C-comodule is a pair $\{(N, \Delta_N) \mid N\text{-vector space}, \Delta_N : N \xrightarrow[\text{linear}]{} C\otimes N\}$ such that
"coaction"

$$\begin{tikzcd} N \arrow[r, "\Delta_N"] \arrow[d, "\Delta_N"'] & C\otimes N \arrow[d, "\Delta\otimes Id"] \\ C\otimes N \arrow[r, "Id\otimes\Delta_N"] & C\otimes C\otimes N \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} k\otimes N & C\otimes N \arrow[l, "\varepsilon\otimes Id"'] \\ & N \arrow[u, "\Delta_N"'] \arrow[ul, "\simeq"] \end{tikzcd} \quad \text{commute.}$$

Def: (a) Given two C-comodules (N, Δ_N), $(N', \Delta_{N'})$ a linear map $C \xrightarrow{f} C'$ is a morphism if

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "\Delta_N"'] & N' \arrow[d, "\Delta_{N'}"] \\ C\otimes N \arrow[r, "Id\otimes f"] & C\otimes N' \end{tikzcd} \quad \text{commutes}$$

(b) A subspace $N' \subset N$ is a subcomodule if $\Delta_N(N') \subset C\otimes N'$.

Similarly to the case of modules, one can define the notion of right C-comodules in the same spirit with $\Delta_N : N \to N\otimes C$.

Rmk: A right C-comodule is the same as a left C^{op}-comodule.

Ex1: A coalgebra C is naturally a C-comodule ($\Delta_C = \Delta$)

Lemma3: (a) If (C, Δ) is a coalgebra and (N, Δ_N) is a comodule over it, then N^* is a right C^*-module via

$$N^*\otimes C^* \xrightarrow[\lambda]{} (C\otimes N)^* \xrightarrow{\Delta_N^*} N^*$$

(b) If A is a fin. dim. algebra and (M, μ_M) is a right A-module, then M^* is a (left) A^*-comodule via

$$M^* \xrightarrow{\mu_M^*} (M\otimes A)^* \xrightarrow[\simeq]{\lambda^{-1}} A^*\otimes M^*$$

Exercise: Check Lemma.

Ex 2: Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra and M, N be H-comodules. Then $M\otimes N$ has a natural H-comodule structure with a coaction given by

$$\Delta_{M\otimes N} : M\otimes N \xrightarrow{\Delta_M \otimes \Delta_N} H\otimes M\otimes H\otimes N \xrightarrow{Id\otimes \tau_{M,H}\otimes Id} H\otimes H\otimes M\otimes N \xrightarrow{\mu\otimes Id_{M\otimes N}} H\otimes M\otimes N$$

▶ The commutativity of

$$\begin{tikzcd} M\otimes N \arrow[r, "\Delta_{M\otimes N}"] \arrow[dr, "\simeq"'] & H\otimes M\otimes N \arrow[d, "\varepsilon\otimes Id_{M\otimes N}"] \\ & k\otimes M\otimes N \end{tikzcd}$$

is clear from such diagrams for M,N and $\varepsilon(xy)=\varepsilon(x)\varepsilon(y)$.

The commutativity of

$$\begin{tikzcd} M\otimes N \arrow[r, "\Delta_{M\otimes N}"] \arrow[d, "\Delta_{M\otimes N}"'] & H\otimes M\otimes N \arrow[d, "Id\otimes \Delta_{M\otimes N}"] \\ H\otimes M\otimes N \arrow[r, "\Delta\otimes Id_{M\otimes N}"] & H\otimes H\otimes M\otimes N \end{tikzcd}$$

is clear from such diagrams for M,N and $\Delta(xy)=\Delta(x)\Delta(y)$ ■

Ex 3: Given a bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ any vector space V is endowed with a *trivial* comodule structure via the unit map

$$\Delta_V : V \simeq k\otimes V \xrightarrow{\eta\otimes Id_V} H\otimes V.$$

Ex 4 (Free comodule): Given a coalgebra (C, Δ, ε) and a vector space V, the free C-comodule on V is $(C\otimes V, \Delta\otimes Id_V)$.

The following result is treated similarly to the case of modules:

Lemma 4: (a) If H is a bialgebra, M,N,P – H-comodules, k – trivial H-comodule, then: $(M\otimes N)\otimes P \simeq M\otimes(N\otimes P)$ and $k\otimes M\simeq M\simeq M\otimes k$ as H-comodules.

(b) If H is cocommutative, then the flip $\tau_{M,N} : M\otimes N \xrightarrow{\sim} N\otimes M$ is an isomorphism of H-comodules.

▶ Obvious ■

We will also use Sweedler's notations for comodules, where we agree to write $\Delta_N(x) = \sum_{(x)} x_c \otimes x_N$.

Then, the two conditions on comodule read as follows:

$$\sum_{(x)} (x_c)'\otimes (x_c)''\otimes x_N = \sum_{x} x_c \otimes (x_N)_C \otimes (x_N)_N$$

$$\sum_{(x)} \varepsilon(x_c)\, x_N = x$$

Finally let us discuss the case when a vector space M is equipped both with an H-module and an H-comodule structures (H-bialgebra)

$$\mu_M : H\otimes M \longrightarrow M \quad \text{and} \quad \Delta_M : M \longrightarrow H \otimes M.$$

Then $H\otimes M$ is an H-module (with the help of Δ) and an H-comodule (with the help of μ).

Lemma 5: TFAE:

(a) μ_M is a morphism of comodules

(b) Δ_M is a morphism of modules.

▶ (a) is equivalent to the commutativity of the following diagram

$$\begin{tikzcd} H\otimes M \arrow[r, "\Delta_{H\otimes M}"] \arrow[d, "\mu_M"'] & H\otimes H\otimes M \arrow[d, "Id\otimes\mu_M"] \\ M \arrow[r, "\Delta_M"] & H\otimes M \end{tikzcd}$$

(b) is equivalent to the commutativity of the following diagram

$$\begin{tikzcd} H\otimes M \arrow[r, "Id\otimes\Delta_M"] \arrow[d, "\mu_M"'] & H\otimes H\otimes M \arrow[d, "\mu_{H\otimes M}"] \\ M \arrow[r, "\Delta_M"] & H\otimes M \end{tikzcd}$$

Comparing these diagrams it suffices to prove $(Id\otimes\mu_M)\Delta_{H\otimes M} = \mu_{H\otimes M}(Id\otimes\Delta_M)$

Choose $x\in H$, $a\in M$. Then $((Id\otimes\mu_M)\Delta_{H\otimes M})(x\otimes a) = \sum_{(x)(a)} x' a_H \otimes \mu_M(x'', a_M)$

$$(\mu_{H\otimes M}(Id\otimes\Delta_M))(x\otimes a) = \sum_{(x)(a)} x' a_H \otimes \mu_M(x'', a_M)$$

Def: We say that M is an *H-bimodule* if the equivalent conditions of Lemma above are satisfied.

Ex 5: Let H be a bialgebra and V a vector space. Then $H\otimes V$ is endowed both with a *trivial* comodule structure and a *trivial* module structure over H (induced by Δ and μ). Then $H\otimes V$ is an H-bimodule.

Our goal for the next few lectures is to introduce $SL_q(2)$ and $U_q(\mathfrak{sl}(2))$.

Today: Hopf algebras $SL(2)$ and $GL(2)$.

General picture

① Given an algebraic monoid with unite: $G\times G \xrightarrow{\mu} G$, $\{e\} \xrightarrow{\eta} G$, we have the induced maps $k[G] \xrightarrow{\Delta} k[G\times G] = k[G]\otimes k[G]$ and $k[G] \xrightarrow{\varepsilon} k[\{e\}] = k$

Exercise: $(k[G], \Delta, \varepsilon)$ is a coalgebra

(follows immediately from associativity of μ and unit condition).

Moreover, we note that $k[G]$ is tautologically a commutative algebra with 1_G (function on G everywhere equal to 1) being a unity.

Exercise: Check that the above structures make $k[G]$ into a bialgebra.

② If in addition, G is a group, with the inverse map $inv: G \to G$, s.t.

$$\begin{tikzcd} G \arrow[r, "diag"] \arrow[rrr, bend right, "g\mapsto e\ \forall g"'] & G\times G \arrow[r, "Id\times inv"'] & G\times G \arrow[r, "\mu"] & G \end{tikzcd}$$

commutes, then the induced diagram on the level of functions is also commutative:

$$\begin{tikzcd} k[G] & k[G\times G]=k[G]\otimes k[G] \arrow[l, "\text{product map}"'] & k[G]\otimes k[G] \arrow[l, "Id\otimes S"'] & k[G] \arrow[l, "\Delta"'] \arrow[lll, bend left, "\eta\varepsilon"] \end{tikzcd} \qquad (*)$$

where $S: k[G] \to k[G]$ is induced by inv.

Upshot: The commutativity of $(*)$ implies that S is an antipode (to be honest, you need to write a similar diagram with $S\otimes Id$ as well)

Thus, $k[G]$ becomes a Hopf algebra.

③ Finally if we have an algebraic action $G \curvearrowright X$ given by $G\times X \xrightarrow{\mu_X} X$, s.t.

$$\begin{tikzcd} G\times G\times X \arrow[r, "Id\times\mu_X"] \arrow[d, "\mu\times Id"] & G\times X \arrow[d, "\mu_X"] \\ G\times X \arrow[r, "\mu_X"] & X \end{tikzcd}$$

is comm., and compatible with unit,

then we get $k[X] \xrightarrow{\Delta_X} k[G\times X] = k[G]\otimes k[X]$, making $k[X]$ into a comodule (e.g. the first condition on comodules follows from above diagram)

Even more: $k[X]$ becomes an $k[G]$-comodule-algebra, see below.

‖ **Def**: Let $(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H)$ be a bialgebra, (A, μ_A, η_A) – an algebra. Then, A is called an H-comodule-algebra if

(a) there is a comodule structure $\Delta_A : A \to H \otimes A$

(b) the maps $\mu_A : A \otimes A \to A$, $\eta_A : A \to k$ are morphisms of H-comodules.

Lemma 6: Given a bialgebra H and an algebra A together with a comodule structure $\Delta_A : A \to H \otimes A$, the condition (b) is equivalent to

(b′) the map $\Delta_A : A \to H \otimes A$ is an algebra morphism

Rmk: In Sweedler's notations (b′) can be written as

$$\sum_{(xy)} (xy)_H \otimes (xy)_A = \sum_{(x)(y)} x_H y_H \otimes x_A y_A, \quad \Delta_A(1) = 1 \otimes 1$$

▶ Similar to our proof of two equivalent conditions on algebra and coalgebra structures on the same space to be compatible – [Lecture 1, Prop 1] ∎

Exercise: Verify that $k[X]$ is a $k[G]$-comodule-algebra.

Key Example to ①-③ is $Mat_2 \curvearrowright k^2$

Let $M(2)$ be an algebra of functions on 2×2 matrices$/k$. As an algebra $M(2) \simeq k[a,b,c,d]$ – polynomial algebra in 4 generators. In particular, for any commutative algebra A, $\mathrm{Hom}(M(2), A) \overset{1-1}{\longleftrightarrow} Mat_2(A)$

$$f \longmapsto \begin{pmatrix} f(a) & f(b) \\ f(c) & f(d) \end{pmatrix}$$

Since $Mat_2(A)$ has a monoid structure, it induces the corresponding coproduct on $M(2)$

$$\boxed{\Delta : M(2) \to M(2) \otimes M(2) \text{ convenient to depict via } \Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

What we actually mean by this is that on generators:

$\Delta(a) = a \otimes a + b \otimes c$, $\Delta(b) = a \otimes b + b \otimes d$, $\Delta(c) = c \otimes a + d \otimes c$, $\Delta(d) = c \otimes b + d \otimes d$.

Note that since $M(2)$ is a polynomial algebra, this assignment determines Δ. Moreover, $Mat_2(A)$ has a unit $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which on the level of $M(2)$ induces

$$\boxed{\varepsilon : M(2) \to k \text{ conveniently depicted by } \varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

Summarizing the general picture ①, we see that $M(2)$ with a tautological algebra structure and Δ, ε defined above is a bialgebra!

However, there is no antipode for $M(2)$. To fix this, we need to consider invertible matrices.

Let $GL(2)$ and $SL(2)$ be the alg-s of functions on the locus of 2×2-matrices consisting of invertible matrices and those with $\det = 1$.

Explicitly: $GL(2) = M(2)[t]/((ad-bc)t-1)$

$SL(2) = M(2)/(ad-bc-1)$

Lemma 7: (a) Formulas $\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\Delta(t) = t\otimes t$

$\varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\varepsilon(t) = 1$

define bialgebra structures on $GL(2)$ and $SL(2)$

(b) The algebra morphism S given by $S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad-bc)^{-1}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$S(t) = t^{-1} = ad-bc$

is an antipode for both of these bialgebras

Conclusion: $M(2)$ – bialgebra, $SL(2)$ & $GL(2)$ – Hopf algebras.

All of them are commutative, but none is cocommutative.

Finally, following the general picture ③, we see that $Mat_2(A) \curvearrowright A^2$ induces a comodule-algebra structure (over $M(2), GL(2), SL(2)$) on $k[x,y] =: A$. Explicitly, the coaction can be conveniently written as $\Delta_A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$, which means that $\Delta_A : A \to M(2)\otimes A$ is s.t.

$\Delta_A(x) = a\otimes x + b\otimes y$

$\Delta_A(y) = c\otimes x + d\otimes y$

Rmk: It is clear from these formulas that each of the graded pieces $k[x,y]_n = \{f \in k[x,y] \mid \text{tot.deg}(f) = n\}$ is a subcomodule.

Def (a) The quantum plane is an algebra $k_q[x,y] := \underbrace{k\langle x,y\rangle}_{\text{free alg.}} / (yx - qxy)$

(b) For any algebra R, an R-point of the quantum plane is a pair $(X, Y) \in R^2$ s.t. $YX = q\cdot XY$